A test of goodness-of-fit for the copula densities

Ghislaine GAYRAUD [*] and Karine TRIBOULEY [♭]

[*] CREST and LMRS-UMR 6085
Université de Rouen
Avenue de l'Université, BP.12
76801 Saint-Étienne-du-Rouvray, FRANCE

Ghislaine.Gayraud@univ-rouen.fr

[♭] LPMA and ModalX
Universite Paris 10
Batiment G
200 avenue de la République
92 001 Nanterre Cedex, FRANCE

karine.tribouley@u-paris10.fr

Abstract

We consider the problem of testing hypotheses on the copula density from n bi-dimensional observations. We wish to test the null hypothesis characterized by a parametric class against a composite nonparametric alternative. Each density under the alternative is separated in the L_2-norm from any density lying in the null hypothesis. The copula densities under consideration are supposed to belong to a range of Besov balls. According to the minimax approach, the testing problem is solved in an adaptive framework: it leads to a $\log \log$ term loss in the minimax rate of testing in comparison with the non-adaptive case. A smoothness-free test statistic that achieves the minimax rate is proposed. The lower bound is also proved. Besides, the empirical performance of the test procedure is demonstrated with both simulated and real data.

Index Terms — Adaptation, Copula Density, Minimax Theory of Test, Goodness Test of Fit.

AMS Subject Classification — 62G10, 62G20, 62G30.

1   Introduction

Copulas became a very popular and attractive tool in the recent literature for modeling multivariate observations. The nice feature of copulas is that they capture the structure dependence among the components of a multivariate observation without requiring the study of the univariate margins. More precisely, Sklar's Theorem ensures that any $d-$varied distribution function H may be expressed as

$$H(x^1, \ldots, x^d) = C\left(F^1(x^1), \ldots, F^d(x^d) \right),$$

where the F^p's are the margins and C is called the copula function. [24] states the existence and the uniqueness of C as soon as the random variables with joint law H are continuous.

Modeling the dependence is a great challenge in statistics, specially in finance or assurance where (for instance) the identification of the dependence structure between assets is essential. Many authors proposed parametrical families of copulas $\{C_\lambda, \lambda \in \Lambda\}$, each of them being available to capture different dependence behavior. The elliptic family contains the Gaussian copulas and the Student copula which are often used in finance. For insurance purposes, heavy tails are needed and copulas coming from the archimedian family are used. Among others, the more common are the Gumbel copula, the Clayton copula or the Frank copula. In view to illustrate the different behaviours of the tails of several copula densities, some graphs corresponding to the models cited above are presented below. The parameters are chosen such a way that the associated Kendall's tau (i.e. the indicator of concordance/discordance) is identical in all illustrations.
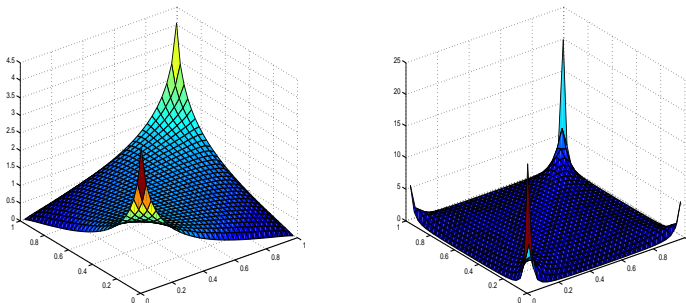

Figure 1: Kendall's tau= 0.25. Left: Bi-dimensional Gaussian copula density with parameter $\rho = 0.4$. Right: Bi-dimensional Student copula density with parameter $(\rho, \nu) = (0.4, 1)$.

Since many parametric copula models are now available, the crucial choice for the practitioner is to identify the model which is well-adapted to data at hand. Many goodness-of-fit tests are proposed in the literature. [14] give an excellent review and propose a detailed empirical study for different tests: we refer to this paper for any supplementary references. Roughly speaking, they study procedures based on empirical processes. Among others, they deal with rank-based versions of the Cramér-von-Mises and Kolmogorov-Smirnov statistics. They also consider test based on Kendall's transform. Basically, they restrict themselves to test statistics built from empirical distributions (empirical copula or transform of this latter). On a theoretical point of view, the asymptotic law under the
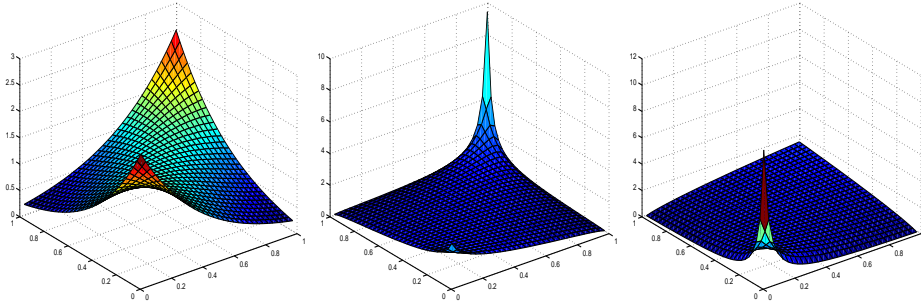
Figure 2: Kendall's tau= 0.25. Left: Bi-dimensional Frank copula density with parameter $\theta = 2.5$. Center: Bi-dimensional Gumbel copula density with parameter $\theta = 1.33$. Right: Bidimensional Clayton copula density with parameter $\theta = 0.66$.

null of the test statistic is stated in a number of papers (see by instance [5], [6] and [7]). It allows in particular to derive the critical value but generally the alternative is unspecified and the properties on the power are empirically given from simulations.

In our paper, it is supposed that the copula C admits a density copula c with respect to the Lebesgue measure. To our knowledge, [8] was the first author to propose a goodness-of-fit test based on nonparametric kernel estimations of the density copula. In the same spirit as the papers cited above, he derived the asymptotic law of the test statistic under the null. His results are valid for bandwidths greater than $n^{-2/(8+d)}$ which correspond to enough smooth copula densities.

Here, we focus on the minimax theory framework: we define the test problem as initiated by [16]. One of the advantages of this point of view is to precisely define the alternative: it is then possible to quantify the risk associated with the test problem as the sum of the first type error and the second type of error. Since this risk measure provides a quality criterion, it is then possible to compare the test procedures. Indeed, the alternative $H_1(v_n)$ is defined from a positive quantity v_n measuring the distance between the null and the latter. Obviously, the larger is this separating distance, the easier is the decision. The aim of the minimax theory is to determine the larger alternative for which the decision remains feasible. Solving **the lower bound problem** is equivalent to exhibit the faster separating rate v_n such that the risk is bounded from below by a given positive constant α: this rate is called **the minimax rate of testing**. Next, **the upper bound problem** has to be solved exhibiting a test procedure whose risk is bounded from above by a given α, that is, the statistic test allows to distinguish the null from $H_1(v_n)$, where v_n is the minimax rate.

In the white noise model or in the density model, the goodness-of-fit problem (stands as explained above) was solved for different regularity classes (Hölder or Sobolev or Besov) associated with various geometries: pointwise, quadratic and supremum norm. For fixed smoothness of the unknown density (**minimax context**), there is a rich literature summed-up in [17] and in [19]. Optimal test procedures include orthogonal projections, kernel estimates or χ^2 procedures. Goodness-of-fit tests with alternatives of variable smoothness into some given interval (**adaptive context**) were introduced by [25] for the

L_2-norm in the Gaussian white noise model and generalized by [26] to L_p-norms. [18] proved that a collection of χ^2 tests attains the adaptive rates of goodness-of-fit tests in L_2-norm as well as for the density model.

For sake of simplicity, we restrict ourselves to bi-dimensional data but there is no theoretical obstacle to generalize our results to higher dimensions. Suppose that we observe n i.i.d. copies $(X_i, Y_i)_{i \in \mathcal{I}}$ where $\mathcal{I} = \{1, \ldots, n\}$ of (X, Y). The random vector (X, Y) is drawn from the distribution function H expressed through the copula C. Moreover, it is assumed that C has a copula density c with respect to the Lebesgue measure and F and G stand for the cdf's of X and Y respectively. From $(X_i, Y_i)_{i \in \mathcal{I}}$, we are interested in studying the goodness-of-fit problem when the null is a composite hypothesis $H_0 : c \in \mathcal{C}_\Lambda$ for a general class $\mathcal{C}_\Lambda$ of parametrical copula densities. Since the alternative is defined from the quadratic distance, we propose a goodness-of-fit test based on wavelet estimation of an integrated functional of the copula density. Indeed, [12] and [1] show that the wavelet methods are an efficient tool to estimate the copula densities since these latter have very specifics behaviors. Unfortunately no direct observations $(F(X_i), G(Y_i))$ for $i \in \mathcal{I}$ are available since F and G are unknown, the test statistic is then built with pseudo-observations $(\widehat{F}(X_i), \widehat{G}(Y_i))_{i \in \mathcal{I}}$: as usual in the copula context, the quantities of interest are rank-based statistics. We provide an auto-driven test procedure and we produce its rate when the alternative contains a regular constraint: since the procedure is based on wavelet methods, the linked functional classes are the Besov classes $B_{s,p,q}$. We give results for $p \geq 2$ (**dense case**) and $s \geq 1/2$. The constraint $s \geq 1/2$ is due to the fact that pseudo-data are used and then a minimal regularity is required in order to pay no attention to substitute the direct data with the ranked data. Observe that [20] have the same constraint in the univariate regression model when the design is random with unknown distribution. Next, we prove that our procedure is minimax (and adaptive) optimal by exhibiting the minimax adaptive rate. This one looks like the minimax rate but an extra $\log \log$ term appears: we prove that this loss is the price to paid for adaptivity. To our knowledge, the proof of the adaptive lower bound in the multivariate density model when the null is composite has never been clearly written.

Next, we allocate a part to empirical studies. Simulation allows us to show that, when the theoretical framework is respected, the power qualities of our test procedures are good. We choose to make simulations starting from the parametrical copula families presented at the beginning of the introduction and which are the more common for applications. We compare our simulation results with those of [14]. Then, we study a very well known sample of real life data of [9] consisting of the indemnity payment (LOSS) and the allocated loss adjustment expense (ALAE) for 1500 general liability claims. The most popular model for the copula is a Gumbel copula model with parameter $\theta = 1.45$ (which may be estimated by inverting the Kendall's tau) given in Figure 3. Among other results, it is empirically shown that the Gumbel and the Gaussian copula models are acceptable while Student, Clayton or Frank models are rejected. Figure 3 gives a wavelet estimator of the copula density of $(LOSS, ALEA)$ by the method explained in [1]. Visually, fitting the unknown copula with the Gumbel model seems indeed to be the most appropriated.

The paper is organized as follows. In Section 2, we first provide a general description of orthonormal wavelet bases, focusing on the mathematical properties that are essential to the construction of the statistics that we consider. In Section 3, we provide the inference procedures: first, we explain how to estimate the square L_2-norm of the copula density
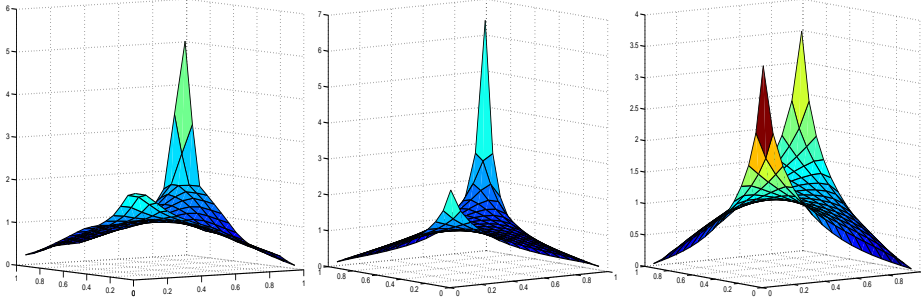
Figure 3: Left: Thresholded wavelet estimator for the copula density of $(LOSS, ALEA)$ as given in [1]. Center: Gumbel copula density with parameter $\theta = 1.45$. Right: Gaussien copula density with parameter $\rho = 0.48$.

and next we derive the procedure of goodness-of-fit. The theoretical part is exposed in Section 4: first, we state very precisely the test problem under consideration; we define the criterion allowing to measure the quality of test procedures and define the separating minimax rate. In Section 5, the main results are stated: our test procedure is shown to be optimal in the sense defined in the previous section. Section 6 is devoted to practical results with both simulated and real data. We conclude these parts with a discussion in Section 7. The proof of the upper bound is given in Section 8 while the proof of the lower bound is given in Section 9. Finally, all technical or computational lemmas which are not essential to understand the main proofs, are postponed in appendices.

2 Wavelet Setting

2.1 Wavelet expansion

In the univariate case, we consider a wavelet basis of $L_2([0,1])$ (see [4]). Let ϕ be the scaling function and let ψ be the same notation for the associated wavelet function and its usual modifications near the frontiers 0 and 1. They are chosen compactly supported on $[0, L]$, $L > 0$. Let j in $\mathbb{N}$, k_1 in $\mathbb{Z}$ and for any univariate function Φ, set $\Phi_{j,k_1}(\cdot) = 2^{j/2}\Phi(2^j \cdot -k_1)$. In the sequel, we use wavelet expansions for bivariate functions and we keep the same notation as for the univariate case. Then, a bivariate wavelet basis is built as follows:

$$\phi_{j,k}(x,y) = \phi_{j,k_1}(x)\phi_{j,k_2}(y), \qquad \psi_{j,k}^{(1)}(x,y) = \phi_{j,k_1}(x)\psi_{j,k_2}(y),$$
$$\psi_{j,k}^{(2)}(x,y) = \psi_{j,k_1}(x)\phi_{j,k_2}(y), \qquad \psi_{j,k}^{(3)}(x,y) = \psi_{j,k_1}(x)\psi_{j,k_2}(y),$$

where the subscript $k = (k_1, k_2)$ indicates the number of components of the functions $\phi_{j,k}$ and $\psi_{j,k}$. For a given $j \in \mathbb{N}$, the set

$$\{\phi_{j,k}, \psi_{\ell,k'}^{\epsilon}, \ \ell \geq j, (k, k') \in \mathbb{Z}^2 \times \mathbb{Z}^2, \epsilon = 1, 2, 3\}$$

is an orthonormal basis of $L^2([0,1]^2)$ and the expansion of any real bivariate function Φ in $L^2([0,1]^2)$ is given by:

$$\Phi(x,y) = \sum_{k\in\mathbb{Z}^2} A_{j,k}\phi_{j,k}(x,y) + \sum_{\ell=j}^{\infty} \sum_{k\in\mathbb{Z}^2} \sum_{\epsilon=1,2,3} B_{\ell,k}^{\epsilon}\psi_{\ell,k}^{\epsilon}(x,y),$$

where the scaling coefficients and the wavelet coefficients are

$$\forall j \in \mathbb{N}, \forall k \in \mathbb{Z}^2, \quad A_{j,k} = \int_{[0,1]^2} \Phi\phi_{j,k}, \qquad B_{j,k}^{\epsilon} = \int_{[0,1]^2} \Phi\psi_{j,k}^{\epsilon}.$$

The Parseval Equality immediately leads to the expansion of the square L_2-norm of the function Φ:

$$\int \Phi^2 = T_j + B_j, \tag{1}$$

where the trend and the detail terms are respectively:

$$T_j = \sum_{k\in\mathbb{Z}^2} (A_{j,k})^2 \text{ and } B_j = \sum_{\ell=j}^{\infty} \sum_{k\in\mathbb{Z}^2} \sum_{\epsilon=1}^{3}(B_{\ell,k}^{\epsilon})^2. \tag{2}$$

Notice that, since the support of Φ is $[0,1]^2$, the sum over the indices k is finite: there are no more than $(2^j + L)^2$ terms in the sum (recall that L is the length of the support of ϕ). In order to simplify the notations, the bounds of variation of k and ϵ in expansion of any Φ, are omitted in the sequel.

2.2 Besov Bodies and Besov spaces

Dealing with wavelet expansions, it is natural to consider Besov bodies as functional spaces since they are characterized in term of wavelet coefficients as follows.

Definition 1. *For any $s > 0$, $p \geq 1$ and any radius $M > 0$, a $d-$varied function Φ belongs to the ball $b_{s,p,\infty}(M)$ of the Besov body $b_{s,p,\infty}$ if and only if its sequence of wavelet coefficients $B_{j,k}^{\epsilon}$ satisfies*

$$\forall j \in \mathbb{N}, \ \sum_{k\in\mathbb{Z}^2} \sum_{\epsilon=1}^{3} |B_{j,k}^{\epsilon}|^p < M2^{-j(s+d/2-d/p)p}.$$

The Besov body $b_{s,p,\infty}$ coincides with the more standard Besov space $\mathcal{B}_{s,p,\infty}$ when there exists an integer N strictly larger than s and such that the $q-$th moment of the wavelet ψ vanishes for any $q = 0, \ldots, N-1$. It is possible to build univariate wavelets whose support is included in $[0, 2N-1]$ satisfying this property for any choice of N (see the Daubechies wavelets).

In the sequel, we need to bound the detail term B_j defined in (2). We use the following inequality

$$\forall j \in \mathbb{N}, \quad B_j \leq \sum_{\ell=j}^{\infty} \left(\sum_{k\in\mathbb{Z}^2} \sum_{\epsilon=1}^{3} |B_{\ell,k}^{\epsilon}|^p \right)^{2/p} \left(K\, 2^{2j} \right)^{1-2/p},$$

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where K is a positive constant depending on the supports of Φ and ψ. Assuming that the function Φ belongs to $b_{s,p,\infty}(M)$ with s, p and M as in Definition 1, the following inequality holds

$$\forall j \in \mathbb{N}, \quad B_j \quad \leq \quad \tilde{K}\, 2^{-2js}, \tag{3}$$

where $\tilde{K}$ is a positive constant depending on the supports of Φ, ψ and on the radius M. When Φ is a copula density, $\tilde{K} = M^{2/p} \left(3(L+1)^2\right)^{1-2/p}$.

3 Statistical Procedures

Assuming that the copula density c belongs to $L_2([0,1]^2)$, we first explain the procedure to estimate the square L_2-norm of c

$$\theta = \|c\|^2 := \int_{[0,1]^2} c^2,$$

which is used to define the alternative of the goodness-of-fit test. The statistical methods depend on parameters (the level j for the estimation procedure and j and the critical value t_j for the test procedure) which are discussed and determined in an optimal way in Section 5.

It is fundamental to Notice that, for any bivariate function Φ, one has

$$\mathbb{E}_c\left[\Phi(U,V)\right] \quad = \quad \mathbb{E}_h\left[\Phi(F(X),G(Y))\right], \tag{4}$$

where h stands for the joint density of (X,Y). This means in particular that the wavelet coefficients $\{c_{j,k}, c_{\ell,k}^{\epsilon},\ \ell \geq j, k \in \mathbb{Z}^2, \epsilon = 1,2,3\}$ of the copula density c on the wavelet basis

$$\{\phi_{j,k}, \psi_{\ell,k}^{\epsilon},\ \ell \geq j, k \in \mathbb{Z}^2, \epsilon = 1,2,3\}$$

are equal to the coefficients of the joint density h on the warped wavelet family

$$\{\phi_{j,k}(F(\cdot),G(\cdot)), \psi_{\ell,k}^{\epsilon}(F(\cdot),G(\cdot)),\ \ell \geq j, k \in \mathbb{Z}^2, \epsilon = 1,2,3\}.$$

The statistical procedures are based on the wavelet expansion of the copula density c, for which the wavelet coefficients have to be estimated.

3.1 Procedures to estimate θ

Let J be a subset of $\mathbb{N}$ and consider a given j in J. Motivated by the wavelet expansion (1), we propose to estimate θ with an estimator of the trend T_j omitting the detail term B_j. Using the orthonormality property of the wavelet basis, it leads to estimate the square of the coefficients of the copula density on the scaling function. As usual, a $U-$statistic associated with the empirical coefficients is used in order to remove the bias terms. Due to (4), we first consider the following family of statistics $\{\widehat{T}_j, j \in J\}$ defined by

$$\widehat{T}_j \quad = \quad \sum_k \widehat{\theta_{j,k}},$$

where $\widehat{\theta_{j,k}}$ is the following $U-$statistic

$$\widehat{\theta_{j,k}} = \frac{1}{n(n-1)} \sum_{\substack{i_1,i_2=1 \\ i_1 \neq i_2}}^{n} \phi_{j,k}\left(F(X_{i_1}), G(Y_{i_1})\right) \phi_{j,k}\left(F(X_{i_2}), G(Y_{i_2})\right).$$

Since no direct observation $(F(X_i), G(Y_i))$ is usually available, it is replaced in $\widehat{\theta_{j,k}}$ by the pseudo observation $(\widehat{F}(X_i), \widehat{G}(Y_i))$, where $\widehat{F}, \widehat{G}$ denote some estimator of the margins. To preserve the independence given by the observations, we split the initial sample $(X_i, Y_i)_{i \in \mathcal{I}}$ into disjoint samples $(X_i, Y_i)_{i \in \mathcal{I}_1}$ and $(X_i, Y_i)_{i \in \mathcal{I}_2}$ with $\mathcal{I}_2 \cup \mathcal{I}_1 = \mathcal{I}$, $\mathcal{I}_2 \cap \mathcal{I}_1 = \emptyset$, and whose size is n_1 and n_2 respectively. The sub-sample with indices in $\mathcal{I}_1$ is used to estimate the marginal distributions and the second one with indices in $\mathcal{I}_2$ is devoted to the computation of the U-statistic. We consider the usual empirical distribution functions:

$$\widehat{F}(x) = \frac{1}{n_1} \sum_{i \in \mathcal{I}_1} \mathbb{1}_{\{X_i \leq x\}} \quad \text{and} \quad \widehat{G}(y) = \frac{1}{n_1} \sum_{i \in \mathcal{I}_1} \mathbb{1}_{\{Y_i \leq y\}}.$$

It leads to the family $\{\widetilde{T}_j, j \in J\}$ of estimators of θ

$$\widetilde{T}_j = \sum_k \widetilde{\theta_{j,k}},$$

with

$$\widetilde{\theta_{j,k}} = \frac{1}{n_2(n_2-1)} \sum_{\substack{i_1,i_2 \in \mathcal{I}_2 \\ i_1 \neq i_2}} \phi_{j,k}\left(\frac{R_{i_1}}{n_1}, \frac{S_{i_1}}{n_1}\right) \phi_{j,k}\left(\frac{R_{i_2}}{n_1}, \frac{S_{i_2}}{n_1}\right),$$

where $R_p = n_1 \widehat{F}(X_p)$ and $S_p = n_1 \widehat{G}(Y_p)$, $p \in \mathcal{I}_1$, could be viewed as estimates of the rank statistics of X_p and Y_p respectively.

3.2 Test Procedures

In this part, we consider a family of known bivariate copula densities $\mathcal{C}_\Lambda = \{c_\lambda, \lambda \in \Lambda\}$ indexed by a parameter λ varying in a given set $\Lambda \subset \mathbb{R}^{d_\Lambda}$, $d_\Lambda \in \mathbb{N}^*$. From the observations $(X_i, Y_i)_{i \in \mathcal{I}}$, our aim is to test the goodness-of-fit between any c_λ and a copula density c, which is *enough distant* in the L_2-norm, from the parametric family $\mathcal{C}_\Lambda$. Acting as in paragraph 3.1, we estimate the square L_2-norm between c and a fixed element c_λ lying in the family $\mathcal{C}_\Lambda$ by

$$\widetilde{T}_j(\lambda) = \sum_k \widetilde{\theta_{j,k}}(\lambda), \tag{5}$$

for

$$\widetilde{\theta_{j,k}}(\lambda) = \frac{1}{n_2(n_2-1)} \sum_{\substack{i_1,i_2 \in \mathcal{I}_2 \\ i_1 \neq i_2}} \left(\phi_{j,k}\left(\frac{R_{i_1}}{n_1}, \frac{S_{i_1}}{n_1}\right) - c_{j,k}(\lambda)\right)$$

$$\times \left(\phi_{j,k}\left(\frac{R_{i_2}}{n_1}, \frac{S_{i_2}}{n_1}\right) - c_{j,k}(\lambda)\right),$$

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where $\{c_{j,k}(\lambda), k \in \mathbb{Z}^2, j \in \mathbb{N}\}$ denote the known scaling coefficients of the target copula density c_λ. Notice that, if direct observations $(F(X_i), G(Y_i))_{i \in \mathcal{I}}$ would be available, the appropriate test statistic $\widehat{T}_j(\lambda)$ would be

$$\widehat{T}_j(\lambda) \;\;=\;\; \sum_k \widehat{\theta}_{j,k}(\lambda),$$

where

$$\widehat{\theta_{j,k}}(\lambda) \;\;=\;\; \frac{1}{n_2(n_2-1)} \sum_{\substack{i_1,i_2 \in \mathcal{I}_2 \\ i_1 \neq i_2}} \left(\phi_{j,k}\left(F(X_{i_1}), G(Y_{i_1})\right) - c_{j,k}(\lambda)\right)$$
$$\times \left(\phi_{j,k}\left(F(X_{i_2}), G(Y_{i_2})\right) - c_{j,k}(\lambda)\right).$$

Now we are ready to build the test procedures. Let us give a set of indices J and a set of critical values $\{t_j, \; j \in J\}$ and define $\{D_j^\Lambda, j \in J\}$, the family of test statistics

$$D_j^\Lambda \;\;=\;\; \mathbb{1}_{\inf_{\lambda \in \Lambda} \widetilde{T}_j(\lambda) > t_j} \quad,$$

allowing to test if c belongs to the parametric family $\mathcal{C}_\Lambda = \{c_\lambda, \lambda \in \Lambda\}$. Notice that $\Lambda = \{\lambda_0\}$ leads to the single null hypothesis $H_0 : c = c_{\lambda_0}$. We are also interested in building auto-driven procedures by considering all the tests in the family

$$D_\Lambda \;\;=\;\; \max_{j \in J} D_j^\Lambda = \mathbb{1}_{\max_{j \in J}(\inf_{\lambda \in \Lambda} \widetilde{T}_j(\lambda) - t_j) > 0} . \tag{6}$$

The sequence of parameters t_j of the method are determined in an optimal way in Section 5. We explain in Section 4 what "optimal way" means in giving a presentation of the minimax theory for our framework.

4 Minimax Theory

We adopt the minimax point of view to solve the problem of hypothesis testing, initiated by [16] in Gaussian white noise. A review of results obtained in problems of minimax hypothesis testing is available in [17] and [19]. Let us describe this approach.

4.1 Minimax hypothesis testing Problem

As in the previous section, we consider $\mathcal{C}_\Lambda = \{c_\lambda, \lambda \in \Lambda\}$ a given functional class of copula densities. For any given $\tau = (s, p, M)$, with $s > 0, p \geq 1, M > 0$, the following statistical problem of hypothesis testing is considered,

$$H_0 : c = c_\lambda \in \mathcal{C}_\Lambda \qquad \text{against} \qquad H_1 : c \in \Gamma(v_n(\tau)), \tag{7}$$

with

$$\Gamma(v_n(\tau)) = b_{s,p,\infty}(M) \cap \left\{c : \inf_{c_\lambda \in \mathcal{C}_\Lambda} \|c - c_\lambda\| \geq v_n(\tau)\right\},$$

where $b_{s,p,\infty}(M)$ is the ball of radius M of the Besov body $b_{s,p,\infty}$ defined in Definition 1 and $v_n(\tau)$ is a sequence of positive numbers, depending on τ and decreasing to zero as n goes to infinity. Recall that $\|g\|$ denotes the L_2-norm of any function g in $L_2([0,1]^2)$. Observe that the functional class $\Gamma(v_n(\tau))$, which determines the alternative H_1, is characterized by three parameters: the regularity class $b_{s,p,\infty}$ where the copula density is supposed to belong, the L_2-norm which is the geometrical tool measuring the distance between both hypotheses, and the sequence $v_n(\tau)$.

According to the principle of the minimaxity, the regularity space and the loss function are chosen by the statistician. Notice that the parameter τ could be known or unknown. Obviously, our aim is to consider tests which are able to detect alternatives defined with sequences $v_n(\tau)$ as small as possible. It can be shown ([17]) that $v_n(\tau)$ cannot be chosen in an arbitrary way: indeed, if $v_n(\tau)$ is too small, then H_0 and H_1 cannot be distinguished with a given error $\alpha \in (0,1)$. Therefore, solving hypothesis testing problems via the minimax approach consists in determining the smallest sequence $v_n(\tau)$ for which such a test is still possible and to indicate the corresponding test functions. The smallest sequence $v_n(\tau)$ is called the minimax rate of testing. Let D_n be a *test statistic* i.e. an arbitrary function with possible values $0, 1$, measurable with respect to $(X_i, Y_i)_{i\in\mathcal{I}}$ and such that we accept H_0 if $D_n = 0$ and we reject it if $D_n = 1$.

Definition 2. *Assuming τ to be known, the sequence $v_n(\tau)$ is the minimax rate of testing H_0 versus H_1 if relations (8) and (9) are fulfilled:*

- *for any given $\alpha_1 \in (0,1)$, there exists $a > 0$ such that*

$$\lim_{n\to+\infty} \inf_{D_n} \left(\sup_{c_\lambda \in \mathcal{C}_\Lambda} I\!P_\lambda(D_n = 1) + \sup_{c\in\Gamma(a\,v_n(\tau))} I\!P_c(D_n = 0) \right) \geq \alpha_1, \qquad (8)$$

  where the infimum is taken over any test statistic D_n,

- *there exists a sequence of test statistics $(D_n^\star)_n$ for which for any given α_2 in $(0,1)$, it exists $A > 0$ such that*

$$\lim_{n\to+\infty} \left(\sup_{c_\lambda \in \mathcal{C}_\Lambda} I\!P_\lambda(D_n^\star = 1) + \sup_{c\in\Gamma(A\,v_n(\tau))} I\!P_c(D_n^\star = 0) \right) \leq \alpha_2, \qquad (9)$$

where $I\!P_c$, respectively $I\!P_\lambda$ denotes the distribution function associated with the copula density c, respectively with c_λ.

4.2 Adaptation

Nevertheless, since the copula function itself is unknown, the a priori knowledge on τ could appear unrealistic. Therefore, the purpose of this paper is to solve the previous problem of test in an adaptive framework i.e. in supposing that $\tau = (s, p, M)$ is unknown but varying in a known set $\mathcal{S}$. Comparing the adaptive case with the non-adaptive case, it has been proved in different frameworks that a loss of efficiency in the rate of testing is unavoidable (see for instance [25], [10]). This loss is expressed as t_n, a positive constant or a sequence of positive numbers increasing to infinity with n (as slow as possible), which appears in the rate of testing $v_{nt_n^{-1}}(\tau)$. Similarly to the minimax rate of testing, we define the adaptive minimax rate of testing as follows.

Definition 3. *The sequence $v_{nt_n^{-1}}(\tau)$ is the adaptive minimax rate of testing if relations (10) and (11) are satisfied*

- *for any given $\alpha_1 \in (0,1)$, there exists $a > 0$ such that*

$$\lim_{n\to+\infty} \inf_{D_n} \left(\sup_{c_\lambda \in \mathcal{C}_\Lambda} I\!\!P_\lambda(D_n = 1) + \sup_{\tau \in \mathcal{S}} \sup_{c \in \Gamma(a\, v_{nt_n^{-1}}(\tau))} I\!\!P_c(D_n = 0) \right) \geq \alpha_1, \qquad (10)$$

  where the infimum is taken over any test statistic D_n,

- *there exists a sequence of universal test statistics $D_n^\star$ (free of τ) such that, for any given α_2 in $(0,1)$, there exists $A > 0$ such that*

$$\lim_{n\to+\infty} \left(\sup_{c_\lambda \in \mathcal{C}_\Lambda} I\!\!P_\lambda(D_n^\star = 1) + \sup_{\tau \in \mathcal{S}} \sup_{c \in \Gamma(A\, v_{nt_n^{-1}}(\tau))} I\!\!P_c(D_n^\star = 0) \right) \leq \alpha_2 \qquad (11)$$

  where t_n is either a positive constant or a sequence of positive numbers increasing to infinity with n as slow as possible.

Notice that relations (10) and (11) (instead of relations (8) and (9)) mean that the minimax rate of testing $v_n(\tau)$ is contaminated by the term t_n in the adaptive setting. Observe that the same phenomenon is observed in the estimation problem where an extra logarithm term $\tilde{t}_n = \log(n)$ has often (but not always) to be paid for the adaptation.

5 Main results

In this section, we focus on test problems for which the parametric family $\mathcal{C}_\Lambda$ is included in some $b_{s_\Lambda, p_\Lambda, \infty}(M_\Lambda)$ where $s_\Lambda > 0$, $p_\Lambda \geq 1$ and $M_\Lambda > 0$ are known.

Our theoretical results concern the minimax resolution of the problem of hypothesis testing defined in (7) in an adaptive framework. Theorem 1 states the result of the lower bound (see relation (10)). Then, Theorem 2 exhibits the rate achieved by the test procedure proposed in Section 3 (see relation (11)). Comparing the rate of our procedure with the fastest rate given in Theorem 1 leads to Theorem 3 establishing the optimality of our procedure.

First, let us state the assumption which gives a control of the complexity of $\mathcal{C}_\Lambda$.

- **A0:** the set Λ is compact in $I\!\!R^{d_\Lambda}$ and

$$\sup_{(x,y)\in[0,1]^2} |c_\lambda(x,y) - c_{\lambda'}(x,y)| \leq Q\|\lambda - \lambda'\|_{I\!\!R^{d_\Lambda}}^\nu, \ \forall \lambda, \ \lambda' \in \Lambda,$$

  where ν is a positive real, Q is a positive constant and $\|\cdot\|_{I\!\!R^{d_\Lambda}}$ denotes the Euclidean norm in $I\!\!R^{d_\Lambda}$.

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5.1 Lower Bound

As it is usual for composite null hypotheses, the result of the lower bound requires the existence of a particular density $c_{\lambda_0} \in \mathcal{C}_\Lambda$ (see assumption **AInf** below) in order to construct a randomized class of functions which must be included in the alternatives.

- **AInf:** there exists a parameter λ_0 in Λ such that

$$\forall (u,v) \in [0,1]^2, \quad c_{\lambda_0}(u,v) > m, \quad \text{with } m > 0.$$

Theorem 1. *Suppose that $\mathcal{S}$ defined by*

$$\mathcal{S} = \{\tau = (s,p,M), s \geq 1/2, p \geq 2, M > 0 : \ s - 2/p \leq s_\Lambda - 2/p_\Lambda, M_\Lambda \leq M\} \qquad (12)$$

is nontrivial (see [25]), which means that there exist $p \geq 2$, $M > 0$ and $0 < s_{min} < s_{max}$ such that

$$\forall s \in [s_{min}, s_{max}], \quad (s,p,M) \in \mathcal{S}$$

*and assume that **A0** and **AInf** hold. Set*

$$v_{nt_n^{-1}}(\tau) = (nt_n^{-1})^{-2s/(4s+2)} \ \text{with} \ \ t_n = \sqrt{\log(\log(n))}.$$

Then, it exists a positive constant a such that

$$\lim_{n \to +\infty} \left(\inf_{D_n} \{ \sup_{\lambda \in \Lambda} I\!\!P_\lambda(D_n = 1) + \sup_{\tau \in \mathcal{S}} \sup_{c \in \Gamma(a \, v_{nt_n^{-1}}(\tau))} I\!\!P_c(D_n = 0) \} \right) = 1, \qquad (13)$$

where the infimum is taken over any test function D_n.

5.2 Upper Bound

Theorem 2 deals with relation (11) which holds for the test statistic D_Λ defined by relation (6) as soon as the parameters of the methods are chosen as follows. The set $J = \{\lfloor j_0 \rfloor, \ldots, \lfloor j_\infty \rfloor\}$ is determined by

$$2^{j_0} = \log(n_2)\log(n_1), \quad 2^{j_\infty} = \left(\frac{n_2}{\log(n_2)} \right)^{1/2} \wedge \left(\frac{n_1}{\log(n_1)} \right)^{1/2 - 1/2q}, \qquad (14)$$

where q is the order of differentiability of the scaling function ϕ. The critical values satisfy

$$\forall j \in J, \quad t_j = 3\mu \frac{2^j}{n_2} \sqrt{\log\log(n_2)}, \qquad (15)$$

where μ is a positive constant such that $\mu > \sqrt{2K_g K_1}$, and K_g and K_1 are positive constants depending on $\|\phi\|_\infty$, $\|c\|_\infty$, $\|c_\lambda\|_\infty$ and the length of the support of ϕ (see Lemma 3).

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Theorem 2. *Let us choose $n_1 = \pi n$ and $n_2 = (1 - \pi)n$ for some π in $(0, 1)$. Assume that the scaling function ϕ is continuously $q-$differentiable for*

$$q \geq \left[1 - \frac{\log\left(\frac{n_2}{\log(n_2)}\right)}{\log\left(\frac{n_1}{\log(n_1)}\right)} \right]^{-1}.$$

Moreover assume that any density c under the alternatives or any c_λ under the null are uniformly bounded. Then, the test statistic D_Λ defined by (6) is such that

$$\lim_{n_1 \wedge n_2 \to +\infty} \sup_{c_\lambda \in \mathcal{C}_\Lambda} I\!\!P_\lambda(D_\Lambda = 1) = 0. \tag{16}$$

Assume that **A0** *holds, then there exists a positive constant A such that*

$$\lim_{n_1 \wedge n_2 \to +\infty} \sup_{\tau \in \mathcal{S}} \sup_{c \in \Gamma(Av_{nt_n^{-1}}(\tau))} I\!\!P_c(D_\Lambda = 0) = 0, \tag{17}$$

where
$$v_{nt_n^{-1}}(\tau) = (n_2 t_{n_2}^{-1})^{-2s/(4s+2)} \text{ and } t_{n_2} = \sqrt{\log(\log(n_2))}.$$

Relation (11) of the upper bound holds since both relations (16) and (17) are satisfied. Notice also that relation (16) indicates that the test statistic D_Λ is asymptotically of any level in $(0, 1)$.

5.3 Optimality

As a corollary of Theorem 1 and Theorem 2, we obtain

Theorem 3. *Under the assumptions of Theorem 1 and Theorem 2, our test procedure defined by Relation (6) is adaptive optimal over the range of parameters $\tau \in \mathcal{S}$ where $\mathcal{S}$ is defined by equation (12).*

6 Practical results

The purpose of this section is to provide several examples to investigate the performances of the test procedure presented in Section 3. This part is not exactly an illustration of the theoretical part since it does not focus on the separating rate between the alternative and the null hypothesis, but it is devoted to the study of our test procedure from a risk point of view. Note also that we do not use exactly the theoretical procedure described in the previous section. As usual for practical purpose, we replace theoretical quantities by more adapted quantities obtained with resampling methods. In the first part, we fix the test level $\alpha = 5\%$ and we study the empirical power function. In the second part, we present an application to some economical series.

6.1 Methodology

On the contrary to the estimation problem, a smooth wavelet is not needed. The test statistic is then computed with the Haar wavelet since it has a small support and then it leads to a fast computation time. The critical value of the test is determined with bootstrap methods: the standard deviation of the test statistic is computed thanks to $N_{boot} = 20$ resampling. The size of the simulated samples is $n = 2048$ which is reasonable for bi-dimensional problems in an asymptotic context. For the real life data example, the number of data is around $n = 4000$. For the simulation part, the empirical level of the test is derived from $N_{MC} = 500$ replications for each test problem.

6.2 Simulations

The setup of our simulations is closely related to the work of [14], except that they consider small samples (of size 150) since their test procedures are based on the empirical copula distribution (and thus generate parametrical rates). To explore various degrees of dependance, three values of Kendal's tau are considered, namely $\tau = 0.25, 0.50, 0.75$ for the following copula families: Clayton, Gumbel, Frank, Normal and Student with four degrees of freedom (df). Calculations are made with the MatLab Sofware. The results of the simulations are presented in Table 1. For an easier reading, the estimated standard errors of the empirical powers are presented in italics. Furthermore, for each testing problem we highlighted the estimated errors of the first type (estimators of $\alpha = 0.05$) using bold characters. In brackets, we give the results obtained by [14] with their test procedures, denoted CvM and built on rank-based versions of the familiar Cramér-von Mises statistics. It would be also possible, if one is interested in, to compare with the different test procedures (based on the empirical copula distribution) proposed also by [14].

Let us now summarize the conclusions made from the simulation results.

- Our test is degenerated: we almost always accept H_0 (when H_0 is true) while the procedure of Genest et al. [14] produces an excellent estimation of the prescribed level α. It is a characteristic of the adaptive minimax procedures.

- For small level of dependence $\tau = 0.25$, our procedure is very competitive and produces (almost) always a better empirical power than the CvM test. The results are spectacular when the fit c_{λ_0} is a Student(4).

- When a large Kendal's tau is considered, our procedure fails when the data are issued from a Clayton copula density. The procedure is not available to recognize a structure of dependence modeled with a Clayton.

- The improvment of our results with respect to the CvM test is decreasing with the Kendal's tau. The CvM test becomes better when the tau is increasing whereas for us it is the opposite.

In conclusion, we recommend the use of our test procedures when the Kendal's tau is not too large since it seems to outperform the existing procedures based on the copula distribution. This situation corresponds to our theoretical setup related to the functional

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spaces in which the unknown copula density is supposed to live. Unfortunately, the practical results do not give hope for using this procedure when the copula densities present high peaks (as it is the case for the Clayton copula density with a large tau).

6.3 Real data

We present now an application to real data of our test procedure. The level of each test (with simple null hypothesis or multivariate null hypothesis) is $\alpha = 5\%$. To obtain the empirical level, $N = 50$ replications of our procedure computed with the half of the available data (chosen randomly) is used. Table 2 gives the empirical probability to reject the null hypothesis and the final decision. "Yes" means that we accept that the structure of dependence belongs to the considered family and "No" that we reject the fitting.

We consider the data of [9], which were also analyzed by [11], [21], [3] and [13], among others. The data consist of the indemnity payment (LOSS) and the allocated loss adjustment expense (ALAE) for 1466 general liability claims.

We consider the following test problems:

$$H_0 : c \in \mathcal{C}_\Lambda$$

where the parametrical family $\mathcal{C}_\Lambda$ is described in Table 2. Since the Kendall's tau computed with the sample is $\tau = 0.31$, we choose an adapted grid of parameters for each parametrical family of copula densities. Next, assuming that the density copula of the data belongs to a fixed parametric family, we estimate the parameter λ

- by $\hat{\lambda}$ in inverting the Kendall's tau (third part of Table 2 where $H_0 : c = c_{\hat{\lambda}}$).

- by $\tilde{\lambda}$ in minimizing the average square error (ASE) computed thanks to the benchmark given in Figure 3 (fourth part of Table 2 where $H_0 : c = c_{\tilde{\lambda}}$). For information, we give the relative ASE computed with $c_{\tilde{\lambda}}$ into brackets.

The various authors who analyzed this data set concluded that the Gumbel copula provides an adequate representation of the underlying dependence structure. The Gumbel parametric family of extreme-value copulas captures the fact that almost all large indemnity payments generate important adjustment expenses (e.g., investigation and legal costs) while the effort invested in the treatment of a small claim is more variable. Accordingly, the copula exhibits positive but asymmetric dependence. Confirming this result, the adaptive method of estimation proposed by [1] provides a benchmark (see Figure 3) for the copula density associated with the data.

7 Discussion

The paper is mainly devoted to construct an optimal procedure for solving a general nonparametric problem of test: both hypotheses are composite, very general parametric family could be considered under the null. Our procedure is proved asymptotically to be adaptive minimax and the minimax separating rate is exhibited over a range of Besov balls.

Thanks to the simulations and a application to real data, our procedure seems to be competitive on the power point of view even if the setting of test under consideration is, in the simulation study, clearly parametric.

It is worthwhile to point out that the copula model requires more regularity (than the usual density model) since the approximation due to the rank-based statistics needs to be accurate enough (see Lemma 4).

One must notice that only copulas densities belonging to **dense Besov spaces** (i.e. defined with a parameter p larger than 2) are under consideration in this paper although several copula densities with a strong dependence structure belong to **sparse Besov spaces** (i.e. defined with a parameter p smaller than 2). As it is illustrated in the simulation study, our test procedure fails for the Clayton copula density with large parameters. This density is suspected to belong to a sparse Besov ball. The study of sparse Besov balls would require the determination of a new test strategy which would lead to another minimax rate of testing: these objectives are beyond those of the present paper and will be explored in a further work since the set of copulas densities contains a number of sparse functions. For sparse Besov balls and in the white noise model for testing the existence of the signal, [22] proved that the minimax testing rate in the sparse and the dense cases is different. They also proved that it is possible to built an adaptive minimax (non linear) procedure of test for the sparse case.

A very close problem is the sample comparison test (problem with two samples). It could be interesting to test if the structure of dependence between a couple of variables $V_1 = (X, Y)$ is the same as for another couple $V_2 = (Z, T)$. This problem of tests could be stated as follows:

$$H_0 : c_{V_1} = c_{V_2} \qquad \text{against} \qquad H_1 : (c_{V_1}, c_{V_2}) \in \Gamma(v_n(\tau)),$$

with

$$\Gamma(v_n) = \{c_{V_1} \in b_{s_1, p_1, \infty}(M_1)\} \cap \{c_{V_2} \in b_{s_2, p_2, \infty}(M_2)\}$$
$$\cap \{(c_{V_1}, c_{V_2}) : \|c_{V_1} - c_{V_2}\| \geq v_n.\}$$

where v_n is the separating rate of both hypotheses. In an analogous way as in Section 3, the rule for the comparison test would be

$$D = \mathbb{1}_{\max_{j \in J}(\sum_k \widetilde{\theta_{j,k}} - t_j) > 0}$$

with

$$\widetilde{\theta_{j,k}} = \frac{1}{n_2(n_2 - 1)} \sum_{\substack{i_1, i_2 \in \mathcal{I}_2 \\ i_1 \neq i_2}} \left(\phi_{j,k} \left(\frac{R_{i_1}^X}{n_1}, \frac{R_{i_1}^Y}{n_1} \right) - \phi_{j,k} \left(\frac{R_{i_1}^Z}{n_1}, \frac{R_{i_1}^T}{n_1} \right) \right)$$
$$\times \left(\phi_{j,k} \left(\frac{R_{i_2}^X}{n_1}, \frac{R_{i_2}^Y}{n_1} \right) - \phi_{j,k} \left(\frac{R_{i_2}^Z}{n_1}, \frac{R_{i_2}^T}{n_1} \right) \right),$$

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where R^X, R^Y, R^Z, R^T are the rank statistics associated with X, Y, Z, T. Using the same tools as in [2], in which the homogeneity in law of the both samples is studied, it is possible to prove that this test is adaptive optimal and that the minimax rate of testing is

$$v_n = \left(\frac{n}{\sqrt{\log(\log(n_2))}} \right)^{-2(s_1 \wedge s_2)/(4(s_1 \wedge s_2)+2)}.$$

Obviously, all these test procedures could be used in the multivariate framework ($d > 2$), but as usual in the nonparametric context, it will provide slower minimax rates of testing.

8 Proof of Theorem 2

Recall that for any given $\lambda \in \Lambda$, $\mathbb{P}_\lambda$ (respectively $\mathbb{P}_c$) denote the distribution associated with density c_λ, respectively with c. In the same spirit, denote also $\mathbb{E}_\lambda$ and Var_λ (respectively $\mathbb{E}_c$ and Var_c) the expectation and the variance with respect to $\mathbb{P}_\lambda$, respectively to $\mathbb{P}_c$. When no index appears in $\mathbb{E}$ or in $\mathbb{P}$ it means that the underlying distribution is either $\mathbb{P}_c$ or $\mathbb{P}_\lambda$.

8.1 Expansion of the statistics of interest

Fix a level j in J. For the test problem, the statistic of interest $\widetilde{T}_j(\lambda)$ (for $\lambda \in \Lambda$) defined in (5) is an estimate of

$$T_j(\lambda) \;=\; \sum_k \theta_{j,k}(\lambda) = \sum_k \left(c_{j,k} - c_{j,k}(\lambda) \right)^2,$$

which is the quantity that we need to detect under the alternative. It would be useful to expand the statistic $\widetilde{T}_j(\lambda)$ as follows

$$\widetilde{T}_j(\lambda) = 2T_j^\diamond(\lambda) + T_j^\heartsuit + T_j^\spadesuit + 2T_j^\clubsuit(\lambda) + T_j(\lambda) \tag{18}$$
$$= 2\sum_k \theta_{j,k}^\diamond(\lambda) + \sum_k \theta_{j,k}^\heartsuit + \sum_k \theta_{j,k}^\spadesuit + 2\sum_k \theta_{j,k}^\clubsuit(\lambda) + \sum_k \theta_{j,k}(\lambda),$$

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where

$$\theta_{j,k}^{\heartsuit} = \frac{1}{n_2(n_2-1)} \sum_{\substack{i_1,i_2 \in \mathcal{I}_2 \\ i_1 \neq i_2}} (\phi_{j,k}(F(X_{i_1}),G(Y_{i_1})) - c_{j,k})$$
$$\times (\phi_{j,k}(F(X_{i_2}),G(Y_{i_2})) - c_{j,k})$$

$$\theta_{j,k}^{\spadesuit} = \frac{1}{n_2(n_2-1)} \sum_{\substack{i_1,i_2 \in \mathcal{I}_2 \\ i_1 \neq i_2}} \left(\phi_{j,k}\left(\frac{R_{i_1}}{n_1},\frac{S_{i_1}}{n_1}\right) - \phi_{j,k}(F(X_{i_1}),G(Y_{i_1})) \right)$$
$$\times \left(\phi_{j,k}\left(\frac{R_{i_2}}{n_1},\frac{S_{i_2}}{n_1}\right) - \phi_{j,k}(F(X_{i_2}),G(Y_{i_2})) \right)$$

$$\theta_{j,k}^{\clubsuit}(\lambda) = \frac{1}{n_2(n_2-1)} \sum_{\substack{i_1,i_2 \in \mathcal{I}_2 \\ i_1 \neq i_2}} \left(\phi_{j,k}\left(\frac{R_{i_1}}{n_1},\frac{S_{i_1}}{n_1}\right) - \phi_{j,k}(F(X_{i_1}),G(Y_{i_1})) \right)$$
$$\times (\phi_{j,k}(F(X_{i_2}),G(Y_{i_2})) - c_{j,k}(\lambda))$$

$$\theta_{j,k}^{\diamond}(\lambda) = \frac{1}{n_2} \sum_{i_1 \in \mathcal{I}_2} (\phi_{j,k}(F(X_{i_1}),G(Y_{i_1})) - c_{j,k})(c_{j,k} - c_{j,k}(\lambda)).$$

The sequence $\{c_{j,k}\}_{j,k}$ denotes the unknown scaling coefficients of the unknown copula density c. Recall that

$$\widehat{T}_j(\lambda) = \sum_k \widehat{\theta_{j,k}}(\lambda),$$

with

$$\widehat{\theta_{j,k}}(\lambda) = \frac{1}{n_2(n_2-1)} \sum_{\substack{i_1,i_2 \in \mathcal{I}_2 \\ i_1 \neq i_2}} (\phi_{j,k}(F(X_{i_1}),G(Y_{i_1})) - c_{j,k}(\lambda))$$
$$\times (\phi_{j,k}(F(X_{i_2}),G(Y_{i_2})) - c_{j,k}(\lambda)).$$

The following lemma gives some evaluation for the first moments of each statistic of interest.

Lemma 1. *Let q be a positive integer and assume that ϕ is continuously $q-$differentiable. Let j be a level smaller than j_∞ defined in (14). Then, it exists some positive constant κ which may depend on ϕ, $\|c\|_\infty$, $\|c_\lambda\|_\infty$ and M such that*

$$\mathbb{E}\widehat{T}_j(\lambda) = T_j(\lambda) \quad \text{and} \quad \mathrm{Var}\widehat{T}_j(\lambda) \leq \kappa \left(\left(\frac{2^j}{n_2}\right)^2 + \left(\frac{2^j}{n_2}\right) T_j(\lambda) \right)$$

$$\mathbb{E}|T_j^{\spadesuit}| \leq \kappa \frac{\log(n_1)}{n_1}$$

$$\mathbb{E}_c|T_j^{\clubsuit}(\lambda)| \leq \kappa \left(\frac{\log(n_1)}{n_1} T_j(\lambda) \right)^{1/2} \quad \text{and} \quad \mathbb{E}_\lambda(T_j^{\clubsuit}(\lambda))^2 \leq \kappa 2^j \left(\frac{\log(n_1)}{n_2 n_1} \right).$$

Using the Bernstein Inequality, we establish the following bound for the deviation of the statistic $T_j^{\diamond}(\lambda)$ under the alternative. The proof is postponed to Appendix B.

Lemma 2. *For any level j, for all $x > 0$*

$$IP_c\left(|T_j^\diamond(\lambda)| \geq x\right) \leq \exp\left(-K\left(\frac{n_2^2 x^2}{n_2 T_j(\lambda) + n_2 x 2^j T_j(\lambda)^{1/2}}\right)\right),$$

where K is a positive constant depending on $L, \|\phi\|_\infty$ and $\|c\|_\infty$.

Using a result from [15], we establish the following bound for the deviation of the U-statistics $\widehat{T}_j(\lambda)$ and $T_j^\heartsuit$. The proof is postponed to Appendix C.

Lemma 3. *For any level j, as soon as $x \geq 2^j n_2^{-1}\sqrt{\log(\log(n_2))}$, for all $\mu > 0$,*

$$IP_\lambda\left(|\widehat{T}_j(\lambda)| > \mu x\right) + IP_c\left(|T_j^\heartsuit| > \mu x\right) \quad \leq \quad K_g(\log(n_2))^{-\delta}$$

for any positive $\delta \leq \mu^2(K_g K_1)^{-1}$, where K_g is an universal positive constant given in [15] and K_1 is a positive constant depending on $L, \|\phi\|_\infty$ and either $\|c_\lambda\|_\infty$ or $\|c\|_\infty$ depending on the underlying distribution i.e. either IP_λ or IP_c.

8.2   Proof of Relation (16) (First type error)

Let us fix $\lambda \in \Lambda$ and set

$$p_\lambda = IP_\lambda\left(\max_{j \in J}\left[\inf_{\lambda' \in \Lambda}\widetilde{T}_j(\lambda') - t_j\right] > 0\right).$$

Notice that under the null

$$T_j^\diamond(\lambda) = T_j(\lambda) = 0 \text{ and } T_j^\heartsuit = \widehat{T}_j(\lambda).$$

Using expansion (18), we get

$$
\begin{aligned}
p_\lambda &\leq \sum_{j \in J} IP_\lambda\left(\inf_{\lambda' \in \Lambda}\widetilde{T}_j(\lambda') > t_j\right) \\
&\leq \sum_{j \in J} IP_\lambda\left(\widetilde{T}_j(\lambda) > t_j\right) \\
&\leq \sum_{j \in J}\left\{IP_\lambda\left(|\widehat{T}_j(\lambda)| > \frac{t_j}{3}\right) + IP_\lambda\left(|T_j^\spadesuit| > \frac{t_j}{3}\right) + IP_\lambda\left(|T_j^\clubsuit(\lambda)| > \frac{t_j}{3}\right)\right\}
\end{aligned}
$$

Due to Lemma 1 and using Markov Inequality, we obtain

$$
\begin{aligned}
p_\lambda &\leq \sum_{j \in J} IP_\lambda\left(|\widehat{T}_j(\lambda)| > \frac{t_j}{3}\right) + \sum_{j \in J}\left\{\frac{IE_\lambda|T_j^\spadesuit|}{(t_j/3)} + \frac{IE_\lambda(T_j^\clubsuit(\lambda))^2}{(t_j/3)^2}\right\} \\
&\leq \sum_{j \in J} IP_\lambda\left(|\widehat{T}_j(\lambda)| > \frac{t_j}{3}\right) \\
&\quad + K\sum_{j \in J}\left\{(t_j/3)^{-1}\frac{\log(n_1)}{n_1} + (t_j/3)^{-2}\left(\frac{2^j\log(n_1)}{n_1 n_2}\right)\right\}.
\end{aligned}
$$

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Notice that $\widehat{T}_j(\lambda)$ is centered under $I\!P_\lambda$, then applying Lemma 3, where t_j is $t_j = 3\mu\,2^j\,n_2^{-1}\sqrt{\log(\log(n_2))}$, the constant μ is defined in (15) and since $\mathrm{card}(J) \le \log(n_2)$, one obtains

$$
\begin{aligned}
p_\lambda &\le K_g\mathrm{card}(J)\,(\log(n_2))^{-\delta} + K\mathrm{card}(J)2^{-j_0}\left(\frac{n_2\log(n_1)}{n_1\sqrt{\log\log(n_2)}}\right)\\[2mm]
&\quad + K\mathrm{card}(J)2^{-j_0}\left(\frac{\log(n_1)n_2^2}{n_2 n_1\sqrt{\log\log(n_2)}}\right)\\[2mm]
&\le K_g(\log(n_2))^{1-\delta} + K2^{-j_0}\left(\frac{\log(n_1)\log(n_2)}{\sqrt{\log\log(n_2)}}\right),
\end{aligned}
$$

where the last inequality holds since δ satisfies $\delta \le \mu^2(2K_gK_1)^{-1}$ (see Lemma 3). Since μ is such that $\mu > \sqrt{2K_gK_1}$, relation (16) is proved if one takes $\delta = \mu^2(2K_gK_1)^{-1}$.

8.3   Proof of Relation (17) (Second type error)

Let us fix $\tau \in \mathcal{S}$ and $c \in \Gamma(Av_{nt_n^{-1}}(\tau))$ and set

$$
p_c = I\!P_c\left(\max_{j\in J}\inf_{\lambda\in\Lambda}\widetilde{T}_j(\lambda) - t_j \le 0\right).
$$

Using the expansion (18), we get, for any $j^\star \in J$

$$
\begin{aligned}
p_c &\le I\!P_c\left(\inf_\lambda\left\{2T_{j^\star}^{\diamond}(\lambda) + T_j(\lambda) + T_{j^\star}^{\heartsuit} + T_{j^\star}^{\spadesuit} + 2T_{j^\star}^{\clubsuit}(\lambda)\right\} \le t_{j^\star}\right)\\[2mm]
&\le I\!P_c\left(\inf_\lambda\left\{2T_{j^\star}^{\diamond}(\lambda) + T_{j^\star}(\lambda)\right\} \le 2t_{j^\star}\right)\\[2mm]
&\quad + I\!P_c\left(T_{j^\star}^{\heartsuit} + T_{j^\star}^{\spadesuit} + 2\inf_\lambda\left\{T_{j^\star}^{\clubsuit}(\lambda)\right\} \ge t_{j^\star}\right)\\[2mm]
&\le I\!P_c\left(\inf_\lambda\left\{2T_{j^\star}^{\diamond}(\lambda) + T_{j^\star}(\lambda)\right\} \le 2t_{j^\star}\right) + I\!P_c\left(T_{j^\star}^{\heartsuit} \ge t_{j^\star}/3\right)\\[2mm]
&\quad + I\!P_c\left(T_{j^\star}^{\spadesuit} \ge t_{j^\star}/3\right) + I\!P_c\left(\inf_\lambda\left\{T_{j^\star}^{\clubsuit}(\lambda)\right\} \ge t_{j^\star}/6\right)\\[2mm]
&= p_{c1}(j^\star) + p_{c2}(j^\star) + p_{c3}(j^\star) + p_{c4}(j^\star).
\end{aligned}
\tag{19}
$$

Let us explain how $j^\star$ is chosen. From the wavelet expansion (1) and Lemma 1, one has

$$
I\!E_c\widehat{T_{j^\star}}(\lambda) = T_{j^\star}(\lambda) = \int (c-c_\lambda)^2 - B_{j^\star}(\lambda),
$$

where $T_{j^\star}$, $B_{j^\star}$ are defined in (2) and $t_j^\star$ is the critical value given in (15). Since c is in $\Gamma(Av_{nt_n^{-1}}(\tau))$ and c_λ lies in $b_{s_\Lambda,p_\Lambda,\infty}(M_\Lambda) \subset b_{s,p,\infty}(M)$, the function $(c-c_\lambda)$ is in $b_{s,p,\infty}(M)$. We can choose $j^\star$ such that

$$
2^{j^\star} = \left(\frac{\tilde{K}}{3\mu}\frac{n_2}{\sqrt{\log\log(n_2)}}\right)^{1/(2s+1)},
$$

which is possible due to our choice of j_∞ and because $s \geq 1/2$; the constant $\tilde{K}$ appears in (3). It implies that $B_{j^\star} \leq t_{j^\star}$ since $B_{j^\star} \leq \tilde{K}2^{-2j^\star s}$ (see Inequality (3)). Next, since $c \in \Gamma(Av_{nt_n^{-1}}(\tau))$, one has $\int(c - c_{\lambda'})^2 \geq A^2(v_{nt_n^{-1}}(\tau))^2$ for all $\lambda' \in \Lambda$. Focusing on rates $v_{nt_n^{-1}}(\tau)$ combined with positive constant A which satisfy $4t_{j^\star} \leq (Av_{nt_n^{-1}}(\tau))^2$, one obtains

$$\frac{t_{j^\star}}{\mathbb{E}_c \widehat{T_{j^\star}}(\lambda)} = \frac{t_{j^\star}}{T_{j^\star}(\lambda)} \leq 1/3. \tag{20}$$

Coming back to the evaluation of the probability terms (see relation (19)), we first consider $p_{c1}(j^\star)$. Consider an η-net Λ_η on the set Λ that is for any λ in Λ, denote $\tilde{\lambda}$ the closest (in the Euclidean sense) element in Λ_η to λ (closer than η). Due to assumption **A0**, let us prove that for any $j \in J$, $T_j^\diamond(\tilde{\lambda}) + T_j(\tilde{\lambda})$ is close to $T_j^\diamond(\lambda) + T_j(\lambda)$:

$$
\begin{aligned}
|T_j^\diamond(\tilde{\lambda}) - T_j^\diamond(\lambda)| &= \left| \sum_k \left[\frac{1}{n_2} \sum_{i_1 \in \mathcal{I}_2} (\phi_{j,k}(F(X_{i_1}), G(Y_{i_1})) - c_{j,k}) \left(c_{j,k}(\tilde{\lambda}) - c_{j,k}(\lambda) \right) \right] \right| \\
&\leq \left[\sum_k \left(c_{j,k}(\tilde{\lambda}) - c_{j,k}(\lambda) \right)^2 \right]^{1/2} \\
&\quad \times \left[\sum_k \left(\frac{1}{n_2} \sum_{i_1 \in \mathcal{I}_2} (\phi_{j,k}(F(X_{i_1}), G(Y_{i_1})) - c_{j,k}) \right)^2 \right]^{1/2} \\
&\leq Q\eta^\nu \, 2^{2j}(2\|\phi\|_\infty + \|c\|_\infty 2^{-4j})^{1/2}.
\end{aligned}
$$

In the same way, one has,

$$|T_j(\lambda) - T_j(\tilde{\lambda})| \leq \kappa Q\eta^\nu \left(2\max(\|c\|, \|c_{\tilde{\lambda}}\|) + Q\eta^\nu \right).$$

Choosing $\eta = n^{-b}$ with $b\nu > 1$, then by (20) and applying Lemma 2, we get

$$
\begin{aligned}
p_{c1}(j^\star) &\leq \sum_{\lambda \in \Lambda_{n^{-b}}} \mathbb{P}_c \left(2T_{j^\star}^\diamond(\lambda) + T_{j^\star}(\lambda) \leq 2t_{j^\star} \right) \\
&\leq \sum_{\lambda \in \Lambda_{n^{-b}}} \mathbb{P}_c \left(T_{j^\star}^\diamond(\lambda) \leq -T_{j^\star}(\lambda)/6 \right) \\
&\leq \sum_{\lambda \in \Lambda_{n^{-b}}} \exp \left[-K \left(\frac{n_2}{2^{2j^\star}} T_{j^\star}(\lambda) \wedge \frac{n_2}{2^{j^\star}} T_{j^\star}(\lambda)^{1/2} \right) \right] \\
&\leq \sum_{\lambda \in \Lambda_{n^{-b}}} \exp \left[-K_2 \left(n_2 t_{j^\star} \wedge \frac{n_2}{2^{j^\star}} t_{j^\star}^{1/2} \right) \right] \\
&\leq \left(\frac{D(\Lambda)}{n^{-b}} \right)^{d_\Lambda} \exp \left[-K_3 \left(2^{j^\star} (\log\log(n_2))^{1/2} \right. \right. \\
&\qquad\qquad\qquad\qquad \left. \left. \wedge \left(\frac{n_2}{2^{j^\star/2}} \right)^{1/2} (\log\log(n_2))^{1/4} \right) \right] \tag{21}
\end{aligned}
$$

where $D(\Lambda)$ is the diameter of Λ and K, K_2 and K_3 are positive constants. Both terms behind the minus sign in the exponential of the right hand side of the last inequality tend

to infinity with a power of n since $s > 1/2$. This implies that $p_{c1}(j^\star)$ goes to zero as n goes to infinity.

Now, it remains to verify that $p_{c2}(j^\star), p_{c3}(j^\star)$ and $p_{c4}(j^\star)$ are going to zero as $n_1 \wedge n_2$ goes to infinity. Using again the bound (20), Lemma 3 for some positive δ, Lemma 1 and the definition of the critical value (15), one gets

$p_{c2}(j^\star) + p_{c3}(j^\star) + p_{c4}(j^\star)$

$$
\begin{aligned}
&\leq \mathbb{P}_c\left(T_{j^\star}^\heartsuit \geq t_{j^\star}/3\right) + + \mathbb{P}_c\left(T_{j^\star}^\spadesuit \geq t_{j^\star}/3\right) + \mathbb{P}_c\left(T_{j^\star}^\clubsuit(\lambda) \geq t_{j^\star}/6\right) \\
&\leq K_g(\log(n_2))^{-\delta} + 9\,\frac{\mathbb{E}_c|T_{j^\star}^\spadesuit(\lambda)|^2}{t_{j^\star}^2} + 6\,\frac{\mathbb{E}_c|T_{j^\star}^\clubsuit|}{t_{j^\star}} \\
&\leq K_g(\log(n_2))^{-\delta} + 9\kappa\frac{2^{j^\star}\log(n_1)}{n_1 n_2\, t_{j^\star}^2} + 6\kappa\frac{\log(n_1)}{n_1\, t_{j^\star}} \\
&\leq K_g(\log(n_2))^{-\delta} + 6\kappa\left(2^{-j^\star}\frac{n_2}{n_1}\frac{\log(n_1)}{\sqrt{\log\log(n_2)}}\right),
\end{aligned}
\tag{22}
$$

which tends to zero with our choice of $j^\star$ and where κ is the positive constant appearing in Lemma 1. Inequalities (21) and (22) entail that the right hand side of (19) is less than any $\alpha \in (0,1)$ as n is large enough. To finish the proof, observe that the choice of $v_{nt_n^{-1}}(\tau)$ is driven by the fact that it corresponds to the smallest sequence such that $4t_j^\star \leq (Av_{nt_n^{-1}}(\tau))^2$, which leads to

$$
v_{nt_n^{-1}}(\tau) \geq \left(2^{j^\star}\mu\frac{\sqrt{\log(\log n_2)}}{n_2}\right)^{1/2} \geq \left(\frac{n_2}{\sqrt{\log\log(n_2)}}\right)^{2s/(4s+2)}.
$$

9 Proof of Theorem 1

Without loss of generality, we suppose that the support of the scaling function ϕ and its associated wavelet function ψ is $[0,1]$. Moreover recall that $\int_0^1 \psi^\epsilon = 0$. Let us give some $a > 0$ which must be small enough.

9.1 Discretisation of $\mathcal{S}$

For any given $\tau = (s, p, M) \in \mathcal{S}$, denote by $j(\tau)$ the level

$$
2^{j(\tau)} = (nt_n^{-1})^{2/(4s+2)}
$$

and define s_j the solution of the equation $j = j(s_j, p, M)$ for any resolution level $j \in \tilde{J} = \{j_{s_{\max}}, \ldots, j_{s_{\min}}\} \subset \{j_0, \ldots, j_\infty\}$ with

$$
j_{s_{\max}} = \lfloor j(s_{\max}, p, M)\rfloor \text{ and } j_{s_{\min}} = \lfloor j(s_{\min}, p, M)\rfloor.
$$

Consider now the set $\mathcal{S}_n = \{\tau_j = (s_j, p, M), j \in \tilde{J}\}$ which appears as a discretisation version of a subset of $\mathcal{S}$ whose cardinality is of order $O(\log(n))$.

9.2 Prior and parametric family included in the alternatives

For any $s_j \in \mathcal{S}_n$, define a prior π_j which is concentrated on the class of the random functions

$$c_j(u,v) = c_{\lambda_0}(u,v) + \sum_k \sum_{\epsilon=1}^{3} \delta_k u_j(n) \psi_{j,k}^{\epsilon}(u,v),$$

where c_{λ_0} is defined in assumption **AInf** and

$$P(\delta_k = 1) = P(\delta_k = -1) = 1/2 \quad \text{and} \quad u_j(n) = C_1 M (n t_n^{-1})^{-\frac{2(s_j+1)}{4s_j+2}}$$

for C_1 such that $3M^2 C_1^2 = 2a^2$. Let j be any index in $\tilde{J}$. Since $\int \psi = 0$ and when a is small enough (to guarantee that $c_j \geq 0$), c_j is a density. Easy calculations imply that

$$\|c_j - c_{\lambda_0}\|^2 = M^2 C_1^2 \, (v_{nt_n^{-1}}(\tau_j))^2 > a^2 \, (v_{nt_n^{-1}})^2.$$

Moreover, if a is small enough, we have $3\, C_1^p < 1$ and

$$2^{j(s_j+1-2/p)p} \sum_k \sum_\epsilon | \int c_j \psi_{jk}^\epsilon |^p \;=\; 2^{j(s_j+1-2/p)p} \sum_k \sum_\epsilon |u_j(n)|^p$$

$$=\; 3 C_1^p M^p \leq M^p,$$

implying that $c_j \in b_{s_j,p,\infty}(M)$. Denote by $\mathcal{A}_{j,n}(a)$ the set of densities

$$\mathcal{A}_{j,n}(a) = \{c \in b_{s_j,p,\infty}(M) : \inf_{\lambda \in \Lambda} \|c - c_\lambda\|^2 > a^2 (v_{nt_n^{-1}}(\tau_j))^2\}.$$

and consider the variation between both distributions $I\!\!P_{\lambda_0}$ and $I\!\!P_\Pi$

$$Var(I\!\!P_{\lambda_0}, I\!\!P_\Pi) = \frac{1}{2} \int \left| \frac{dI\!\!P_\Pi}{dI\!\!P_{\lambda_0}} - 1 \right| dI\!\!P_{\lambda_0},$$

where

$$\frac{dI\!\!P_\Pi}{dI\!\!P_{\lambda_0}} = \frac{1}{N_n} \sum_{j \in \tilde{J}} \frac{dI\!\!P_j}{dI\!\!P_{\lambda_0}} = \frac{1}{N_n} \sum_{j \in \tilde{J}} I\!\!E_{\pi_j}^{(n)} [\frac{c_j}{c_{\lambda_0}}],$$

and $N_n = \operatorname{card}(\tilde{J})$. Assuming that the following assertion holds

$$\lim_{n \to \infty} \inf_{j \in \tilde{J}} \pi_j(c \in \mathcal{A}_{j,n}(a)) = 1, \tag{23}$$

we deduce that the left hand side (LHS) of relation (13) without the limit is bounded from below by

$$\begin{aligned} LHS &\geq I\!\!P_{\lambda_0}(D_n = 1) + \sup_{\tau_j \in \mathcal{S}_n} \sup_{c \in \mathcal{A}_{j,n}(a)} I\!\!P_c(D_n = 0) \\ &\geq 1 - Var(I\!\!P_{\lambda_0}, I\!\!P_\Pi)(1 + o_n(1)), \end{aligned}$$

as n large enough. Since the supports of the functions c_j and $c_{j'}$ are disjoint for $j \neq j'$, one has

$$\begin{aligned} 1 - Var(I\!\!P_{\lambda_0}, I\!\!P_\Pi) &\geq 1 - \frac{1}{2} \frac{1}{N_n^2} \sum_{j \in \tilde{J}} I\!\!E_{\lambda_0} \left[\left(\int \prod_{i=1}^{n} \frac{c_j(U_i, V_i)}{c_{\lambda_0}(U_i, V_i)} d\pi_j(c_j) \right)^2 - 1 \right] \\ &\geq 1 - o_n(1) \end{aligned}$$

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provided that

$$\lim_{n\to\infty} \frac{1}{N_n^2} \sum_{j\in\tilde{J}} I\!\!E_{\lambda_0}\left[\left(\int \prod_{i=1}^{n} \frac{c_j(U_i,V_i)}{c_{\lambda_0}(U_i,V_i)} d\pi_j(c_j)\right)^2\right] = 0. \tag{24}$$

Relation (13) is thus proved if (23) and (24) are satisfied. The remaining proofs are given in the sequel.

9.3 Proof of Relation (23)

Let Λ' be a subsect of Λ. We have

$$\pi_j\left(\inf_{\lambda\in\Lambda} \|c_j - c_\lambda\|^2 \le a^2(v_{nt_n^{-1}}(\tau_j))^2\right) \le \pi_j\left(\inf_{\lambda\in\Lambda/\Lambda'} \|c_j - c_\lambda\|^2 \le a^2(v_{nt_n^{-1}}(\tau_j))^2\right)$$

$$+ \pi_j\left(\inf_{\lambda\in\Lambda'} \|c_j - c_\lambda\|^2 \le a^2(v_{nt_n^{-1}}(\tau_j))^2\right) \tag{25}$$

Consider the particular subset Λ' defined by

$$\Lambda' = \{\lambda\in\Lambda: \|c_{\lambda_0} - c_\lambda\|^2 \le 6C_1^2 M^2(v_{nt_n^{-1}}(\tau_j))^2\}.$$

Notice that

$$\lambda\in\Lambda/\Lambda' \implies \|c_\lambda - c_j\|^2 \ge a^2(v_{nt_n^{-1}}(\tau_j))^2$$

due to the choice of C_1. It implies that the first term in the right hand side of (25) is null and then, it remains to prove that

$$\lim_{n\to\infty} \pi_j\left(\inf_{\lambda\in\Lambda'} \|c_j - c_\lambda\|^2 \le a^2(v_{nt_n^{-1}}(\tau_j))^2\right) = 0. \tag{26}$$

Since λ is in Λ', we get

$$\|c_\lambda - c_j\|_2^2 = \|c_{\lambda_0} - c_\lambda\|^2 + \sum_k \sum_\epsilon u_j(n)^2 + 2\sum_k \sum_\epsilon \delta_k u_j(n) B_{j,k,\lambda,\lambda_0}$$

$$\ge 3C_1^2 M^2 (v_{nt_n^{-1}}(\tau_j))^2 + 2\sum_k \delta_k u_j(n) \sum_\epsilon B_{j,k,\lambda,\lambda_0},$$

where

$$B_{j,k,\lambda,\lambda_0} = \int \psi_{j,k}^\epsilon (c_{\lambda_0} - c_\lambda).$$

Therefore assertion (26) is equivalent to

$$\lim_{n\to\infty} \pi_j\left(\inf_{\lambda\in\Lambda'} 2\sum_k \delta_k u_j(n) B_{j,k,\lambda,\lambda_0} \le -a^2(v_{nt_n^{-1}}(\tau_j))^2\right) = 0.$$

or

$$\lim_{n\to\infty} \pi_j\left(\sup_{\lambda\in\Lambda'} 2\sum_k (-\delta_k) u_j(n) B_{j,k,\lambda,\lambda_0} \ge a^2(v_{nt_n^{-1}}(\tau_j))^2\right) = 0. \tag{27}$$

We can construct in the Euclidean metric an η-net Λ'_η on the subset Λ'. For any λ in Λ', denote $\tilde{\lambda}$ the closest element in Λ'_η to λ in the Euclidean sense. Then for any $\lambda \in \Lambda'$, we have by assumption **A0**:

$$
\begin{aligned}
|\sum_k \delta_k u_j(n)(B_{j,k,\lambda,\lambda_0} - B_{j,k,\tilde{\lambda},\lambda_0})| &\leq u_j(n)\sum_k |B_{j,k,\lambda,\lambda_0} - B_{j,k,\tilde{\lambda},\lambda_0}| \\
&\leq u_j(n)\sum_k Q\eta^\nu 2^{-j}\|\psi^\epsilon\|_\infty \\
&\leq 2^{j(s+1)}2^{2j}Q\eta^\nu 2^{-j}\|\psi^\epsilon\|_\infty \\
&\leq \kappa 2^{js}\eta^\nu,
\end{aligned}
$$

where κ is a positive constant depending on Q, C_1, M and $\|\psi^\epsilon\|_\infty$. Chosing $\eta = n^{-b}$, with $b\nu > \frac{s_{\max}}{2s_{\max}+1}$, the proof of relation (27) is then reduced to the proof of

$$
\lim_{n\to\infty} \operatorname{Card}(\Lambda'_{n^{-b}})\pi_j\left(2\sum_k(-\delta_k)u_j(n)B_{j,k,\tilde{\lambda},\lambda_0} \geq a^2(v_{nt_n^{-1}}(\tau_j))^2\right) = 0
$$

$$
\lim_{n\to\infty} (Tn^b)^{d_\Lambda}\pi_j\left(2\sum_k(-\delta_k)u_j(n)B_{j,k,\tilde{\lambda},\lambda_0} \geq a^2(v_{nt_n^{-1}}(\tau_j))^2\right) = 0, \quad (28)
$$

where Diam is the diameter of Λ. Finally, relation (26) is proved applying Bernstein inequality in the right hand side of relation (28). Indeed Bernstein inequality is applied to

$$
\pi_j\left(2\sum_k(-\delta_k)u_j(n)B_{j,k,\tilde{\lambda},\lambda_0} \geq a^2(v_{nt_n^{-1}}(\tau_j))^2\right),
$$

with the i.i.d. centered random variables $Z_k = -\delta_k B_{j,k,\tilde{\lambda},\lambda_0}$. In particular, Notice that $|Z_k| < K_1 v_{nt_n^{-1}}(\tau_j)$, $\sum_k \operatorname{Var}(Z_k) \leq K_2(v_{nt_n^{-1}}(\tau_j))^2$, where K_1 and K_2 are positive constants. Notice also that it leads to an exponential bound of order $\exp(-2^j)$.

9.4 Proof of Relation (24)

Set

$$
l_{n,\pi} = \int \prod_{i=1}^n \frac{c_j(U_i,V_i)}{c_{\lambda_0}(U_i,V_i)}d\pi_j(c_j).
$$

Due to the fact that the functions $\psi^\epsilon_{j,k}$ have disjoint support, it is possible to rewrite c_j as follows

$$
c_j = c_{\lambda_0}\prod_k(1 + \delta_k D_{j,k})
$$

for

$$
D_{j,k} = u_j(n)\sum_\epsilon \frac{\psi^\epsilon_{j,k}}{c_{\lambda_0}}.
$$

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Then,

$$
\begin{aligned}
l_{n,\pi} &= \prod_k \int \prod_{i=1}^n (1 + \delta_k D_{j,k}(U_i, V_i)) d\pi_j(\delta_k) \\
&= \prod_k \frac{1}{2} \left\{ \prod_{i=1}^n (1 + D_{j,k}(U_i, V_i)) + \prod_{i=1}^n (1 - D_{j,k}(U_i, V_i)) \right\},
\end{aligned}
$$

and

$$
\begin{aligned}
l_{n,\pi}^2 &= \prod_k \frac{1}{4} \left\{ 2 \prod_{i=1}^n \left[1 + D_{j,k}^2(U_i, V_i) \right] + 2 \prod_{i=1}^n \left[1 - D_{j,k}^2(U_i, V_i) \right] \right. \\
&\quad \left. + H \left(D_{j,k}(U_i, V_i), \left(D_{j,k}^{b_t}(U_t, V_t) \right)_{t \in \{1,\ldots,i-1,i+1,\ldots,n\}} \right) \right\},
\end{aligned}
$$

where b_t is either 0 or 2. Due to the independence of the data and acting as in [23], it can be shown that

$$
\mathbb{E}_{\lambda_0} \left[H \left(D_{j,k}(U_i, V_i), \left(D_{j,k}^{b_t}(U_t, V_t) \right)_{t \in \{1,\ldots,i-1,i+1,\ldots,n\}} \right) \right] = 0.
$$

Therefore,

$$
\begin{aligned}
\mathbb{E}_{\lambda_0}[l_{n,\pi}^2(U_i, V_i)] &\leq \prod_k \left\{ \left(1 + \mathbb{E}_{\lambda_0} D_{j,k}^2(U_i, V_i) \right)^n + \left(1 - \mathbb{E}_{\lambda_0} D_{j,k}^2(U_i, V_i) \right)^n \right\} \\
&\leq \prod_k \cosh \left(n \mathbb{E}_{\lambda_0} D_{j,k}^2(U_i, V_i) \right).
\end{aligned}
$$

Using the inequality $\log(\cosh(u)) \leq K u^2$ where K is a fixed constant and since c_{λ_0} is bounded from below by m, one obtains

$$
\begin{aligned}
\frac{1}{N_n^2} \sum_{j \in \tilde{J}} \exp(\log(\mathbb{E}_{\lambda_0} l_{n,\pi})^2) &\leq \frac{1}{N_n^2} \sum_{j \in \tilde{J}} \exp \left\{ K n^2 \sum_k \left(\mathbb{E}_{\lambda_0} D_{j,k}^2(U_i, V_i) \right)^2 \right\} \\
&\leq \frac{1}{N_n^2} \sum_{j \in \tilde{J}} \exp \left\{ \frac{3^2 K}{m^2} n^2 2^{2j} u_j(n)^4 \right\} \\
&\leq \frac{\log(n)^\kappa}{\log(n)(1 + o_n(1))},
\end{aligned}
$$

where $\kappa = K(3 C_1^2 M^2)^2 m^{-2} = 4K a^4 m^{-2}$. Choosing a small enough and $\kappa < 1$, Relation (24) is then proved.

10 Appendix A: Proof of Lemma 1

In this part, κ denotes any positive constant which may depend on ϕ, M and on $\|c\|$, $\|c_\lambda\|$.

10.1 Notations and Preliminaries

Let us define or recall some notations that will be used below. For any $k \in \mathbb{Z}^2$, set

$$\xi_k(X_i, Y_i) = \phi_{j,k}\left(\widehat{F}(X_i), \widehat{G}(Y_i)\right) - \phi_{j,k}\left(F(X_i), G(Y_i)\right)$$
$$\omega_{j,k}^\lambda(X_i, Y_i) = \phi_{j,k}(F(X_i), G(Y_i)) - c_{j,k}(\lambda)$$
$$\omega_{j,k}^\infty(X_i, Y_i) = \phi_{j,k}(F(X_i), G(Y_i)) - c_{j,k},$$

where i is in $\mathcal{I}_2$. First, the localization property of the scaling function implies that only few $\xi_k(X_i, Y_i)$ will be used since the others are zero. Indeed, one has the following result

Lemma 4. *For any $k \in \mathbb{Z}^2$, let us denote*

$$N_j = \text{card} \ \{i \in \mathcal{I}_2; \xi_k(X_i, Y_i) \neq 0\}.$$

Let $\delta > 0$. For any level j such that

$$2^j \leq \frac{2}{3\sqrt{\delta+1}} \left(\frac{n_2}{\log(n_2)}\right)^{1/2},$$

one has

$$\mathbb{P}(N_j > 2(2L+3)n_2 2^{-j})) \leq K(n_1^{-\delta} + n_2^{-\delta}).$$

We refer to [12] for the proof of this lemma since a similar result is established with an estimate $\widehat{F}$ built on the whole sample: it guarantees in particular that $\widehat{F}(X_{(i:n)}) = i/n$, where $X_{(i:n)}$ denotes the i-th (among n) order statistic. In our case, the situation is different since $\widehat{F}(X_{(i:n_2)})$ is based on the observations lying in the subsample whose indices are in $\mathcal{I}_1$ whereas it is calculated in an observation lying in the subsample whose indices are in $\mathcal{I}_2$; nevertheless, applying the Dvoretsky–Kiefer–Wolfovitz Inequality, the following deviation inequality holds. For any $\epsilon > 0$, $\mathbb{P}_{\widehat{F}} = \mathbb{P}\left(\left|\widehat{F}(X_{(i:n_2)}) - \frac{i}{n_2}\right| \geq 2\epsilon\right)$ is bounded from above by

$$\mathbb{P}_{\widehat{F}} \leq \mathbb{P}\left(\left|\widehat{F}(X_{(i:n_2)}) - F(X_{(i:n_2)})\right| \geq \epsilon\right) + \mathbb{P}\left(\left|F(X_{(i:n_2)}) - \widetilde{F}(X_{(i:n_2)})\right| \geq \epsilon\right)$$
$$\leq \mathbb{P}\left(\|\widehat{F} - F\|_\infty \geq \epsilon\right) + \mathbb{P}\left(\|\widetilde{F} - F\|_\infty \geq \epsilon\right)$$
$$\leq K\left(n_1^{-\delta} + n_2^{-\delta}\right),$$

as soon as we take $\epsilon = \sqrt{\delta \log(n_1)/(2n_1)} \vee \sqrt{\delta \log(n_2)/(2n_2)}$. Here $\widehat{F}$ represents the empirical margin computed with the subsample whose indices in $\mathcal{I}_1$ and $\widetilde{F}$, the empirical margin computed with the subsample whose indices in $\mathcal{I}_2$.

10.1.1 Study of $\widehat{T}_j(\lambda)$

Rewrite $\widehat{\theta_{j,k}}(\lambda)$ in $\widehat{T}_j(\lambda) = \sum_k \widehat{\theta_{j,k}}(\lambda)$ as follows

$$\widehat{\theta_{j,k}}(\lambda) \;=\; \frac{1}{n_2(n_2-1)} \sum_{\substack{i_1,i_2 \in \mathcal{I}_2 \\ i_1 \neq i_2}} \omega_{j,k}^{\lambda}(X_{i_1},Y_{i_1})\omega_{j,k}^{\lambda}(X_{i_2},Y_{i_2}).$$

For all $i \in \mathcal{I}_2$, one has $I\!\!E(\omega_{j,k}^{\lambda}(X_i,Y_i)) = c_{j,k} - c_{j,k}(\lambda)$, which implies that

$$I\!\!E(\widehat{T}_j(\lambda)) = \sum_k \theta_{j,k}(\lambda) = T_j(\lambda).$$

Moreover for $p \neq k$, one obtains
$$I\!\!E(\widehat{\theta_{j,k}}(\lambda)\widehat{\theta_{j,p}}(\lambda))$$

$$= \; \frac{1}{(n_2(n_2-1))^2} \sum_{i_1 \neq i_2 \neq i_3 \neq i_4} I\!\!E\left[\omega_{j,k}^{\lambda}(X_{i_1},Y_{i_1})\right] I\!\!E\left[\omega_{j,p}^{\lambda}(X_{i_3},Y_{i_3})\right] I\!\!E\left[\omega_{j,k}^{\lambda}(X_{i_2},Y_{i_2})\right] I\!\!E\left[\omega_{j,p}^{\lambda}(X_{i_4},Y_{i_4})\right]$$

$$+4\frac{1}{(n_2(n_2-1))^2} \sum_{i_1 \neq i_2 \neq i_3} I\!\!E\left[\omega_{j,k}^{\lambda}(X_{i_1},Y_{i_1})\right] I\!\!E\left[\omega_{j,p}^{\lambda}(X_{i_3},Y_{i_3})\right] I\!\!E\left[\omega_{j,k}^{\lambda}(X_{i_2},Y_{i_2})\omega_{j,p}^{\lambda}(X_{i_2},Y_{i_2})\right]$$

$$+2\frac{1}{(n_2(n_2-1))^2} \sum_{i_1 \neq i_2} I\!\!E\left[\omega_{j,k}^{\lambda}(X_{i_1},Y_{i_1})\omega_{j,p}^{\lambda}(X_{i_1},Y_{i_1})\right] I\!\!E\left[\omega_{j,k}^{\lambda}(X_{i_2},Y_{i_2})\omega_{j,p}^{\lambda}(X_{i_2},Y_{i_2})\right]$$

$$\leq \; \theta_{j,k}(\lambda)\theta_{j,p}(\lambda) + \frac{4}{n_2}\,(c_{j,k}-c_{j,k}(\lambda))\,(c_{j,p}-c_{j,p}(\lambda)) \left[\int \left(\phi_{j,k} - \int \phi_{j,k}c_\lambda\right)\left(\phi_{j,p} - \int \phi_{j,p}c_\lambda\right)c\right]$$

$$+\frac{2}{n_2(n_2-1)}\left[\int \left(\phi_{j,k} - \int \phi_{j,k}c_\lambda\right)\left(\phi_{j,p} - \int \phi_{j,p}c_\lambda\right)c\right]^2,$$

which implies that

$$\text{Var}(\widehat{T}_j(\lambda)) \;=\; I\!\!E\left(\sum_k \widehat{\theta_{j,k}}(\lambda)\right)^2 - \left(I\!\!E\sum_k \widehat{\theta_{j,k}}(\lambda)\right)^2$$

$$\leq \; \frac{4}{n_2}\sum_{k,p}(c_{j,k}-c_{j,k}(\lambda))\,(c_{j,p}-c_{j,p}(\lambda))\left[\int \left(\phi_{j,k} - \int \phi_{j,k}c_\lambda\right)\left(\phi_{j,p} - \int \phi_{j,p}c_\lambda\right)c\right]$$

$$+\frac{2}{n_2(n_2-1)}\sum_{k,p}\left[\int \left(\phi_{j,k} - \int \phi_{j,k}c_\lambda\right)\left(\phi_{j,p} - \int \phi_{j,p}c_\lambda\right)c\right]^2.$$

Applying the Hölder Inequality and the consequence of the Parseval Equality, we get

$$\sum_{kp}\left[\int \left(\phi_{j,k} - \int \phi_{j,k}c_\lambda\right)\left(\phi_{j,p} - \int \phi_{j,p}c_\lambda\right)c\right]^2$$

$$\leq \; 2^2\left(\sum_{k,p}\left[\int \phi_{j,k}\phi_{j,p}c\right]^2 + 2\sum_{k,p}\left[\int \phi_{j,k}c_\lambda \int \phi_{j,p}c\right]^2 + \left(\sum_k \left[\int \phi_{j,k}c_\lambda\right]^2\right)^2\right)$$

$$\leq \; 2^2\left(\left(\sum_k \int \phi_{j,k}^2 c\right)^2 + 2\int c^2 \int c_\lambda^2 + \left(\int c_\lambda^2\right)^2\right) \leq \kappa\, 2^{2j}.$$

We conclude that

$$\text{Var}(\widehat{T}_j(\lambda)) \;\leq\; \kappa\left(\frac{4}{n_2}\left(\sum_{k,p}\theta_{j,k}(\lambda)\theta_{j,p}(\lambda)\right)^{1/2} 2^j + \frac{2^{2j}}{n_2(n_2-1)}\right)$$

$$\leq \; \kappa\left(\frac{2^j}{n_2}T_j(\lambda) + \frac{2^{2j}}{n_2^2}\right),$$

which is the announced result for $\widehat{T}_j(\lambda)$.

10.1.2 Study of $T_j^\spadesuit$ and $T_j^\clubsuit(\lambda)$

Let us denote

$$
\begin{aligned}
A_{i_1} &= [\xi_k(X_{i_1}, Y_{i_1})], \quad D_{i_1} = \sum_{k,p} \left(I\!E\left[\xi_k(X_{i_1}, Y_{i_1})\xi_p(X_{i_1}, Y_{i_1})\right]\right)^2 \\
B_{i_1,i_2} &= \sum_k [\xi_k(X_{i_1}, Y_{i_1})\xi_k(X_{i_2}, Y_{i_2})], \quad C_{i_1,i_2} = \xi_k(X_{i_1}, Y_{i_1})\xi_p(X_{i_2}, Y_{i_2}).
\end{aligned}
$$

We need the following results which are stated in the lemma below

Lemma 5. *Assume that the scaling function is q-differentiable. For any level $j \leq j_\infty$, there exists some positive constant κ depending on ϕ, its derivatives and on $\|c\|_\infty$ (which might be $\|c_\lambda\|_\infty$ for some $\lambda \in \Lambda$) such that for any distinct indices i_1, i_2, one obtains*

$$
I\!E|A_{i_1}| \leq \kappa \left(\frac{\log(n_1)}{n_1} \right)^{1/2} \tag{29}
$$

$$
I\!E|B_{i_1,i_2}| \leq \kappa 2^{2j} \left(\frac{\log(n_1)}{n_1} \right), \quad I\!E|C_{i_1,i_2}| \leq \kappa \left(\frac{\log(n_1)}{n_1} \right) \tag{30}
$$

$$
|D_{i_1}| \leq 2^{6j} \left(\frac{\log(n_1)}{n_1} \right)^2.
$$

We prove relation (29) in the next section, relations (30) are proven in [12]. We have

$$
I\!E T_j^\spadesuit = \frac{1}{n_2(n_2-1)} \sum_{\substack{i_1,i_2 \in \mathcal{I}_2 \\ i_1 \neq i_2}} I\!E[B_{i_1,i_2}].
$$

Using Lemma 4 and Lemma 5, it follows

$$
I\!E|T_j^\spadesuit| \leq \frac{1}{n_2(n_2-1)} (n_2 2^{-j})^2 \, 2^{2j} \left(\frac{\log(n_1)}{n_1} \right) \leq \left(\frac{\log(n_1)}{n_1} \right).
$$

Moreover, we get

$$
T_j^\clubsuit(\lambda) = \frac{1}{n_2(n_2-1)} \sum_{\substack{i_1,i_2 \in \mathcal{I}_2 \\ i_1 \neq i_2}} \sum_k \left[\xi_k(X_{i_1}, Y_{i_1})\omega_{j,k}^\lambda(X_{i_2}, Y_{i_2}) \right].
$$

By Hölder Inequality and from lemmas 4 and 5, one obtains

$$
I\!E|T_j^\clubsuit(\lambda)| \leq \frac{1}{n_2(n_2-1)} \sum_{\substack{i_1,i_2 \in \mathcal{I}_2 \\ i_1 \neq i_2}}^{n_2} \left(\sum_k (I\!E(A_i))^2 \sum_k (I\!E\omega_{j,k}^\lambda(X_{i_2}, Y_{i_2}))^2 \right)^{1/2}.
$$

Remembering that $\mathbb{E}\omega_{j,k}^{\lambda}(X_{i_2}, Y_{i_2}) = (c_{j,k} - c_{j,k}(\lambda))$ for any index i_2, we get

$$
\begin{aligned}
\mathbb{E}|T_j^{\clubsuit}(\lambda)| &\leq \frac{1}{n_2(n_2-1)}(n_2 2^{-j})n_2 \left[2^{2j}\frac{\log(n_1)}{n_1}T_j(\lambda) \right]^{1/2} \\
&\leq K \left(\frac{\log(n_1)}{n_1}T_j(\lambda) \right)^{1/2}.
\end{aligned}
$$

Let us study the moments of $T_j^{\clubsuit}(\lambda)$ under $\mathbb{P}_\lambda$. Since $\mathbb{E}_\lambda \omega_{j,k}^{\lambda}(X_i, Y_i) = 0$ for any k and i, we obviously have $\mathbb{E}_\lambda T_j^{\clubsuit}(\lambda) = 0$ and

$$
\mathbb{E}_\lambda (T_j^{\clubsuit}(\lambda))^2 = \left(\frac{1}{n_2(n_2-1)} \right)^2 \sum_{i_1 \neq i_2} T_{i_1, i_2} + \left(\frac{1}{n_2(n_2-1)} \right)^2 \sum_{i_1 \neq i_2 \neq i_3} S_{i_1, i_2, i3},
$$

where

$$
\begin{aligned}
T_{i_1, i_2} &= \sum_{k,p} \left(\mathbb{E}_\lambda \left[\xi_k(X_{i_1}, Y_{i_1}) \xi_p(X_{i_1}, Y_{i_1}) \right] \mathbb{E}_\lambda \left[\omega_{j,k}^{\lambda}(X_{i_2}, Y_{i_2}) \omega_{j,p}^{\lambda}(X_{i_2}, Y_{i_2}) \right] \right), \\
S_{i_1, i_2, i_3} &= \sum_{k,p} \left(\mathbb{E}_\lambda \left[\xi_k(X_{i_1}, Y_{i_1}) \xi_p(X_{i_2}, Y_{i_2}) \right] \mathbb{E}_\lambda \left[\omega_{j,k}^{\lambda}(X_{i_3}, Y_{i_3}) \omega_{j,p}^{\lambda}(X_{i_3}, Y_{i_3}) \right] \right).
\end{aligned}
$$

By Hölder Inequality, we have

$$
\begin{aligned}
T_{i_1, i_2} &= \sum_{k,p} \mathbb{E}_\lambda \left[\xi_k(X_{i_1}, Y_{i_1}) \xi_p(X_{i_1}, Y_{i_1}) \right] \mathbb{E}_\lambda \left[\omega_{j,k}^{\lambda}(X_{i_2}, Y_{i_2}) \omega_{j,p}^{\lambda}(X_{i_2}, Y_{i_2}) \right] \\
&\leq D_{i_1}^{1/2} \left(\sum_{k,p} \left(\mathbb{E}_\lambda \left[\omega_{j,k}^{\lambda}(X_{i_2}, Y_{i_2}) \omega_{j,p}^{\lambda}(X_{i_2}, Y_{i_2}) \right] \right)^2 \right)^{1/2}
\end{aligned}
$$

With Parseval Equality, we get

$$
\begin{aligned}
\sum_{k,p} \left(\mathbb{E}_\lambda \left[\omega_{j,k}^{\lambda}(X_{i_2}, Y_{i_2}) \omega_{j,p}^{\lambda}(X_{i_2}, Y_{i_2}) \right] \right)^2 &\leq \sum_{k,p} \left(\int \phi_{j,k} \phi_{j,p} c_\lambda \right)^2 \\
&\leq K \sum_k \int \phi_{j,k}^2 c_\lambda^2 \leq K \, 2^{2j},
\end{aligned}
$$

which combining with Lemma 5, implies that

$$
T_{i_1, i_2} \leq K \left(2^{6j} \left(\frac{\log(n_1)}{n_1} \right)^2 \right)^{1/2} (2^{2j})^{1/2} \leq 2^{4j} \left(\frac{\log(n_1)}{n_1} \right).
$$

In the same way,

$$
\begin{aligned}
S_{i_1, i_2, i_3} &\leq K \left(\sum_{k,p} (\mathbb{E}_\lambda C_{i_1, i_2})^2 \right)^{1/2} \left(\sum_{k,p} \left(\mathbb{E}_\lambda \left[\omega_{j,k}^{\lambda}(X_{i_2}, Y_{i_2}) \omega_{j,p}^{\lambda}(X_{i_2}, Y_{i_2}) \right] \right)^2 \right)^{1/2} \\
&\leq (2^{2j})^{1/2} \left(2^{4j} \left(\frac{\log(n_1)}{n_1} \right)^2 \right)^{1/2} \leq 2^{3j} \left(\frac{\log(n_1)}{n_1} \right).
\end{aligned}
$$

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From Lemma 4, one has

$$
\begin{aligned}
I\!\!E_\lambda(T_j^{\clubsuit}(\lambda))^2 \;\leq\;& K\,\frac{1}{n_2^2(n_2-1)^2}(n_22^{-j})n_22^{4j}\left(\frac{\log(n_1)}{n_1}\right)\\
&+K\,\frac{1}{n_2^2(n_2-1)^2}(n_22^{-j})^2n_22^{3j}\left(\frac{\log(n_1)}{n_1}\right)\\
\leq\;& K\,2^j\left(\frac{\log(n_1)}{n_2n_1}\right).
\end{aligned}
$$

10.2   Proof of Lemma 5

The following expansion is crucial because it allows to reduce the study to univariate variables.

$$
\begin{aligned}
\xi_k(X_i,Y_i) =\;& \xi_{k_1}(X_i)\xi_{k_2}(Y_i) \\
&+\xi_{k_1}(X_i)\phi_{jk_2}\left(G(Y_i)\right)+\xi_{k_2}(Y_i)\phi_{jk_1}\left(F(X_i)\right),
\end{aligned}
\tag{31}
$$

where the univariate statistics $\xi_{k_1}(X_i)$ and $\xi_{k_2}(Y_i)$ are defined as follows

$$
\begin{aligned}
\xi_{k_1}(X_i) &= \phi_{j,k_1}\left(\frac{\widehat{F}(X_i)}{n_1}\right)-\phi_{j,k_1}(F(X_i))\\
\xi_{k_2}(Y_i) &= \phi_{j,k_2}\left(\frac{\widehat{G}(Y_i)}{n_1}\right)-\phi_{j,k_1}(G(Y_i)).
\end{aligned}
$$

Assuming that ϕ is continuously $q-$differentiable, we get

$$
\xi_{k_1}(X_i) \;=\; \hat{z}_{k_1}(X_i)+\hat{w}_{k_1}(X_i),
$$

where

$$
\hat{z}_{k_1}(X_i) \;=\; \sum_{\ell=1}^{q-1}\frac{2^{j\ell}}{\ell!}(\widehat{F}(X_i)-F(X_i))^\ell\,\phi_{j,k_1}^{(\ell)}(F(X_i))
$$

and

$$
\hat{w}_{k_1}(X_i) \;=\; 2^{qj}\int_{\widehat{F}(X_i)}^{F(X_i)}\phi_{j,k_1}^{(q)}(t)\,(F(X_i)-t)^{q-1}dt.
$$

A direct application of the Dvoretsky, Kiefer and Wolfovitz Inequality leads to the following bound

$$
I\!\!P(\|\widehat{F}-F\|_\infty>\epsilon)\leq K\,\exp(-2n_1\epsilon^2)\leq Kn_1^{-\delta},
$$

as soon as $\epsilon=\sqrt{0.5\,\delta\log(n_1)/n_1}$. In the sequel, we take such an ϵ with δ large enough. Since $j\leq j_\infty$ where j_∞ is defined in (14), observe that $2^j\epsilon\leq 1$ and then we get

$$
\begin{aligned}
|\hat{z}_{k_1}(X_i)| &\leq K\,2^j\epsilon\max_{\ell=1,\ldots q-1}|\phi_{j,k_1}^{(\ell)}(F(X_i))|(1+o_P(1))\\
|\hat{w}_{k_1}(X_i)| &\leq K\,2^{(q+1/2)j}\epsilon^q(1+o_P(1))
\end{aligned}
$$

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which leads to the following bound

$$|\xi_{k_1}(X_i)| \leq K \left(2^{(q+1/2)j} \epsilon^q + 2^j \epsilon \max_{\ell=1,\dots q-1} |\phi_{j,k_1}^{(\ell)}(F(X_i))| \right) (1 + o_P(1)).$$

The same kind of result obviously holds for $\xi_{k_2}(Y_i)$. In the sequel, we need the following evaluations (which also hold for any derivatives of ϕ). Using expansion (31), we get

$$\xi_k(X_i, Y_i) = S_1 + S_2,$$

where

$$
\begin{aligned}
S_1 &= \xi_{k_1}(X_i)\xi_{k_2}(Y_i), \\
S_2 &= \xi_{k_1}(X_i)\phi_{j,k_2}(G(Y_i)) + \xi_{k_2}(Y_i)\phi_{j,k_1}(F(X_i)).
\end{aligned}
$$

Using (32), we get

$$
\begin{aligned}
\mathbb{E}|S_1| &\leq K \left(2^{(2q+1)j} \epsilon^{2q} + 2^{(q+1)j} \epsilon^{q+1} + 2^j \epsilon^2 \right), \\
\mathbb{E}|S_2| &\leq K \left(2^{qj} \epsilon^q + \epsilon \right).
\end{aligned}
$$

If $2^j \leq (n_1/\log(n_1))^{1/2-1/2q}$, we obtain $\mathbb{E}|\xi_k(X_i, Y_i)| \leq \epsilon$ which ends the proof.

11 Appendix B : Proof of Lemma 2

Let us denote $T_j^\diamond(\lambda) = n_2^{-1} \sum_{i \in \mathcal{I}_2} Z_i$ where

$$
\begin{aligned}
Z_i &= \sum_k \left(\phi_{jk}(F(X_i), G(Y_i)) - c_{jk} \right) (c_{jk} - c_{jk}(\lambda)), \\
\mathbb{E}_c Z_i &= 0, \\
|Z_i| &\leq \left(\sum_k (\phi_{jk}(F(X_i), G(Y_i)) - c_{jk})^2 \sum_k (c_{jk} - c_{jk}(\lambda))^2 \right)^{1/2} \\
&\leq K_1 \left(\left(2^j\right)^2 T_j(\lambda) \right)^{1/2} \leq K \, 2^j T_j(\lambda)^{1/2},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Var}_c(Z_i) &\leq \sum_{k,p} \mathbb{E} \left(\phi_{jk}(F(X_i), G(Y_i)) - c_{jk} \right) \left(\phi_{jp}(F(X_i), G(Y_i)) - c_{jp} \right) \\
&\qquad \times |(c_{jk} - c_{jk}(\lambda))(c_{jp} - c_{jp}(\lambda))| \\
&\leq \sum_{k,p} \left(\mathbb{E}\phi_{jk}^2(F(X_i), G(Y_i)) \mathbb{E}\phi_{jp}^2(F(X_i), G(Y_i)) \right)^{1/2} \\
&\qquad \times |(c_{jk} - c_{jk}(\lambda))(c_{jp} - c_{jp}(\lambda))| \\
&\leq \|c\|_\infty \sum_k (c_{jk} - c_{jk}(\lambda))^2 = \|c\|_\infty T_j(\lambda).
\end{aligned}
$$

Applying Bernstein Inequality to the Z_i' s leads to prove Lemma 2.

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12 Appendix C: Proof of Lemma 3

12.1 U-Statistic

Let us first recall the result of [15].

Proposition 1. (THEOREM 3.3 P. 21 [15])
It exists an universal positive constant $K_g < \infty$ such that, if Ω is a bounded canonical kernel of two variables for the i.i.d. Z_{i_1}, Z_{i_2}, i_1, $i_2 \in \{1, \ldots, \tilde{n}\}$, where $\tilde{n} \in \mathbb{N}$, for any $x > 0$, we have

$$
I\!P\left(|\sum_{i_1, i_2} \Omega(Z_{i_1}, Z_{i_2})| > x \right) \leq K_g \exp\left(-\frac{1}{K_g} \min\left\{ \frac{x^2}{C^2}, \frac{x}{D}, \left(\frac{x}{B}\right)^{2/3}, \left(\frac{x}{A}\right)^{1/2} \right\} \right),
$$

where

$$
\begin{aligned}
A &= \|\Omega(\cdot, \cdot)\|_\infty, \; B^2 = \tilde{n} \left[\|I\!E[\Omega^2(Z_1, \cdot)]\|_\infty + \|I\!E[\Omega^2(\cdot, Z_2)]\|_\infty \right], \\
C^2 &= \tilde{n}^2 I\!E[(\Omega(Z_1, Z_2))^2]
\end{aligned}
$$

and

$$
D = \tilde{n} \sup_{\Omega_1, \Omega_2} \{ I\!E[\Omega(Z_1, Z_2)\Omega_1(Z_1)\Omega_2(Z_2)] : I\!E[\Omega_1^2(Z_1)] \leq 1; I\!E[\Omega_2^2(Z_2)] \leq 1 \}.
$$

We apply this proposition for $Z_i = (F(X_i), G(Y_i))$, $\tilde{n} = n_2$ and the kernel

$$
\Omega_{\tilde{c}}(Z_{i_1}, Z_{i_2}) = \sum_k \{ \phi_{j,k}(Z_{i_1}) - I\!E_{\tilde{c}}[\phi_{j,k}(Z_{i_1})] \} \times \{ \phi_{j,k}(Z_{i_2}) - I\!E_{\tilde{c}}[\phi_{j,k}(Z_{i_2})] \},
$$

which is considered under the distribution $I\!P_{\tilde{c}}$ where $\tilde{c}$ is either c_λ or c. The quantities A, B, C and D are evaluated in the following lemma which is proved in the next section.

Lemma 6. *There exists some positive constant K_1 larger than either*

$$
\left(12L^2 \|\phi\|_\infty^2 \right) \vee \left(2\|\tilde{c}\|_\infty \right) \vee \left(2L^2 \|\phi\|_\infty^2 \right) \vee \left(4\|\tilde{c}\|_\infty (\|\tilde{c}\|_\infty + 3L^4 \|\phi\|_\infty^2) \right)
$$

such that

$$
A \leq K_1 2^{2j}, \; B^2 \leq K_1 n_2 2^{2j}, \; C^2 \leq K_1 n_2^2 2^{2j}, \; D \leq K_1 n_2,
$$

where $\tilde{c}$ is either c_λ or c.

Again define $\tilde{c}$ as c_λ or c, then applying both the result of [15] and Lemma 6, for any level j and any $x \geq 2^j ((n_2 - 1)n_2)^{-1/2} \sqrt{\log(\log(n_2))}$, it immediately follows that

$$
I\!P_{\tilde{c}}\left(|\frac{1}{n_2(n_2 - 1)} \sum_{\substack{i_1, i_2 \in \mathcal{I}_2 \\ i_1 \neq i_2}} \Omega_{\tilde{c}}(Z_{i_1}, Z_{i_2})| > \mu x \right) \leq K_g \exp(-\delta \log(\log(n_2))).
$$

which ends the proof of Lemma 3.

12.2 Proof of Lemma 6

Let us denote $(U, V) = (F(X), G(Y))$ any pair of random variables whose marginal distribution are both uniform on $[0, 1]$. Denote $\tilde{c}$ the copula density which is c_λ or c; in the same spirit, the coefficients $\tilde{c}_{j,k}$ stand for $c_{j,k}(\lambda)$ or $c_{j,k}$. Recall that

$$
\begin{aligned}
c_{j,k}(\lambda) &= \mathbb{E}_\lambda \left[\phi_{j,k}(F(X), G(Y)) \right] = \int c_\lambda(u,v) \phi_{j,k}(u,v) \, du \, dv. \\
c_{j,k} &= \mathbb{E} \left[\phi_{j,k}(F(X), G(Y)) \right] = \int c(u,v) \phi_{j,k}(u,v) \, du \, dv.
\end{aligned}
$$

Notice that

$$
\sum_{k,p} \mathbb{E}_{\tilde{c}} [\phi_{j,k}(U_{i_1}, V_{i_1}) \phi_{j,p}(U_{i_1}, V_{i_1})] \leq 2^{2j},
$$

$$
\sum_k (\mathbb{E}_{\tilde{c}}[\phi_{j,k}(U,V)])^2 = \sum_k \tilde{c}_{j,k}^2 \leq \|\tilde{c}\|^2 \leq M.
$$

We get

$$
\begin{aligned}
A &= \| \sum_k \left(\phi_{j,k}(u_1, v_1) - \mathbb{E}_{\tilde{c}}[\phi_{j,k}(U,V)] \right) \left(\phi_{j,k}(u_2, v_2) - \mathbb{E}_{\tilde{c}}[\phi_{j,k}(U,V)] \right) \|_\infty \\
&\leq \| \sum_k \phi_{j,k}(u_1, v_1) \phi_{j,k}(u_2, v_2) \|_\infty + 2 \| \sum_k \phi_{j,k}(u_1, v_1) \mathbb{E}_{\tilde{c}}[\phi_{j,k}(U,V)] \|_\infty \\
&\quad + \| \sum_k \left(\mathbb{E}_{\tilde{c}}[\phi_{j,k}(U,V)] \right)^2 \|_\infty \\
&\leq L^2 \, 2^{2j} \|\phi\|_\infty^2 + 2L^2 \|\phi\|_\infty \|\tilde{c}\|_2 2^j + \|\tilde{c}\|_2^2 \leq K 2^{2j},
\end{aligned}
$$

where $K \geq 2L^2 \|\phi\|_\infty^2$ and

$$
\begin{aligned}
B^2 &= 2n_2 \left\| \sum_{k,p} \mathbb{E}_{\tilde{c}} \left[(\phi_{j,k}(U_{i_1}, V_{i_1}) - \mathbb{E}_{\tilde{c}}[\phi_{j,k}(U,V)]) (\phi_{j,p}(U_{i_1}, V_{i_1}) - \mathbb{E}_{\tilde{c}}[\phi_{j,p}(U,V)]) \right] \right. \\
&\qquad \times \left. (\phi_{j,k}(u_2, v_2) - \mathbb{E}_{\tilde{c}}[\phi_{j,k}(U,V)]) (\phi_{j,p}(u_2, v_2) - \mathbb{E}_{\tilde{c}}[\phi_{j,p}(U,V)]) \right\|_\infty \\
&\leq 2n_2 \left\| \sum_{k,p} \left| \int \phi_{j,k} \phi_{j,p} \tilde{c} - \int \phi_{j,k} \tilde{c} \int \phi_{j,p} \tilde{c} \right| (\phi_{j,k}(u_2, v_2) - \mathbb{E}_{\tilde{c}}[\phi_{j,k}(U,V)]) \right. \\
&\qquad \times \left. (\phi_{j,p}(u_2, v_2) - \mathbb{E}_{\tilde{c}}[\phi_{j,p}(U,V)]) \right\|_\infty \\
&\leq 2n_2 (2\|\tilde{c}\|_\infty) \left[\left\| \sum_{k,p} \phi_{j,k}(u_2, v_2) \phi_{j,p}(u_2, z_2) \right\|_\infty + 2 \left\| \sum_{k,p} \phi_{j,k}(u_2, v_2) \mathbb{E}_{\tilde{c}}[\phi_{j,k}(U,V)] \right\|_\infty \right. \\
&\qquad + \left. \left\| \sum_{k,p} \mathbb{E}_{\tilde{c}}[\phi_{j,k}(U,V)] \mathbb{E}_{\tilde{c}}[\phi_{j,p}(U,V)] \right\|_\infty \right] \\
&\leq (4n_2 \|\tilde{c}\|_\infty) \left(2^{2j} L^4 2 \|\phi\|_\infty^2 + 2L^2 2^j \|\phi\|_\infty + 2^{2j} \|\tilde{c}\|_\infty \right) \leq K \, n_2 2^{2j}
\end{aligned}
$$

where $K \geq 4\|\tilde{c}\|_\infty(\|\tilde{c}\|_\infty + 3L^4\|\phi\|_\infty^2)$. Moreover,

$$
\begin{aligned}
C^2 &= n_2^2 \sum_{k,p} \mathbb{E}_{\tilde{c}}\left[(\phi_{j,k}(U_{i_1}, V_{i_1}) - \mathbb{E}_{\tilde{c}}[\phi_{j,k}(U,V)])(\phi_{j,p}(U_{i_1}, V_{i_1}) - \mathbb{E}_{\tilde{c}}[\phi_{j,p}(U,V)])\right] \\
&\quad \times \mathbb{E}_{\tilde{c}}\left[(\phi_{j,k}(U_{i_2}, V_{i_2}) - \mathbb{E}_{\tilde{c}}[\phi_{j,k}(U,V)])(\phi_{j,p}(U_{i_2}, V_{i_2}) - \mathbb{E}_{\tilde{c}}[\phi_{j,p}(U,V)])\right] \\
&= n_2^2 \sum_{k,p} \left(\mathbb{E}_{\tilde{c}}[\phi_{j,k}(U_{i_1}, V_{i_1})\phi_{j,p}(U_{i_1}, V_{i_1})] - \mathbb{E}_{\tilde{c}}[\phi_{j,k}(U,V)]\,\mathbb{E}_{\tilde{c}}[\phi_{j,p}(U,V)]\right)^2 \\
&= n_2^2 \sum_{k,p} \left(\int \phi_{j,k}\phi_{j,p}\,\tilde{c} - \int \phi_{j,k}\,\tilde{c} \int \phi_{j,p}\,\tilde{c}\right)^2 \\
&\leq n_2^2 \sum_{k,p} \left(\int \phi_{j,k}\phi_{j,p}\,\tilde{c}\right)^2 + n_2^2 \left(\sum_k \left(\int \phi_{j,k}\,\tilde{c}\right)^2\right)^2 \\
&\leq n_2^2 \sum_k \int \phi_{j,k}^2\,\tilde{c}^2 + n_2^2 \left(\int \tilde{c}^2\right)^2 \\
&\leq \|\tilde{c}\|_\infty^2 n_2^2 2^{2j} + n_2^2 \|\tilde{c}\|_2^4 \leq K\,n_2^2 2^{2j},
\end{aligned}
$$

where $K \geq 2\|\tilde{c}\|_\infty^2$. Denote $u_{\Omega_1,\Omega_2} = \mathbb{E}_{\tilde{c}}[\Omega_{\tilde{c}}(Z_1, Z_2)\Omega_{1,\tilde{c}}(Z_1)\Omega_{2,\tilde{c}}(Z_2)]$ and for $i = 1, 2$, put

$$
c_i(k) = \int (\phi_{j,k} - \mathbb{E}\tilde{c}\phi_{j,k}(U,V))\Omega_{i,\tilde{c}}\,\tilde{c}.
$$

By Hölder Inequality, we get

$$
\begin{aligned}
u_{\Omega_1,\Omega_2} &= \sum_k \left(\int (\phi_{j,k} - \mathbb{E}_{\tilde{c}}\phi_{j,k}(U,V))\Omega_{1,\tilde{c}}\,\tilde{c}\right)\left(\int (\phi_{j,k} - \mathbb{E}_{\tilde{c}}\phi_{j,k}(U,V))\Omega_{2,\tilde{c}}\,\tilde{c}\right) \\
&\leq \sqrt{\sum_k (c_1(k))^2 \sum_k (c_2(k))^2}.
\end{aligned}
$$

Applying again the inequality of Hölder to $\sum_k (c_1(k))^2$ (the same occurs for $c_2(k)$), one gets

$$
\begin{aligned}
\sum_k (c_1(k))^2 &\leq \sum_k \left(\int (\phi_{j,k} - \mathbb{E}_{\tilde{c}}\phi_{j,k}(U,V))\Omega_{1,\tilde{c}}\,\tilde{c}\,\mathbb{1}_{[\frac{k_1}{2^j},\frac{2N-1+k_1}{2^j}]\times[\frac{k_2}{2^j},\frac{2N-1+k_2}{2^j}]}\right)^2 \\
&\leq \sum_k \left(\int (\phi_{j,k} - \mathbb{E}_{\tilde{c}}\phi_{j,k}(U,V))^2\tilde{c}\right) \times \\
&\qquad \left(\int (\Omega_{1,\tilde{c}})^2\,\mathbb{1}_{[\frac{k_1}{2^j},\frac{2L-1+k_1}{2^j}]\times[\frac{k_2}{2^j},\frac{2L-1+k_2}{2^j}]}\,\tilde{c}\right) \\
&\leq \|\tilde{c}\|_\infty \int (\Omega_{1,\tilde{c}})^2\,\tilde{c} \sum_k \mathbb{1}_{[\frac{k_1}{2^j},\frac{2L-1+k_1}{2^j}]\times[\frac{k_2}{2^j},\frac{2L-1+k_2}{2^j}]} \\
&\leq 12\|\phi\|_\infty^2 L^2,
\end{aligned}
$$

since $\mathbb{E}_{\tilde{c}}(\Omega_{1,\tilde{c}}(U))^2) \leq 1$. It follows that $D \leq K\,n_2$, where $K > 12L^2\|\phi\|_\infty^2$.

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Copula under H_0	True copula	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$
Gumbel	Clayton	1.00 *0.0000* (0.72)	1.00 *0.0000* (0.99)	1.00 *0.0000* (1.00)
	Gumbel	**0.05** *0.0105* **(0.05)**	**0.01** **0.0049** **(0.05)**	**0.00** *0.0000* **(0.05)**
	Frank	0.31 *0.0207* (0.15)	0.36 *0.0214* (0.40)	0.24 *0.0191* (0.84)
	Normal	0.25 *0.0195* (0.10)	0.25 *0.0194* (0.18)	0.09 *0.0127* (0.61)
	Student(4)	0.99 *0.0035* (0.14)	0.89 *0.0139* (0.22)	0.62 *0.0218* (0.55)
Clayton	Clayton	**0.05** *0.0101* **(0.05)**	**0.51** *0.0224* **(0.05)**	**1.00** *0.0000* **(0.05)**
	Gumbel	0.99 *0.0028* (0.86)	1.00 *0.0000* (1.00)	1.00 *0.0000* (1.00)
	Frank	0.97 *0.0079* (0.56)	1.00 *0.0000* (0.96)	1.00 *0.0000* (1.00)
	Normal	0.77 *0.0190* (0.50)	0.85 *0.0178* (0.93)	1.00 *0.0000* (1.00)
	Student(4)	0.45 *0.0223* (0.56)	0.14 *0.0157* (0.95)	1.00 *0.0000* (1.00)
Frank	Clayton	0.99 *0.0022* (0.40)	1.00 *0.0000* (0.89)	1.00 *0.000* (0.97)
	Gumbel	0.19 *0.0175* (0.33)	0.23 *0.0190* (0.63)	0.22 *0.0184* (0.82)
	Frank	**0.05** *0.0108* **(0.05)**	**0.01** *0.0035* **(0.05)**	**0.00** *0.000* **(0.05)**
	Normal	0.14 *0.0155* (0.08)	0.39 *0.0218* (0.20)	0.72 *0.0201* (0.41)
	Student(4)	0.95 *0.0096* (0.18)	0.83 *0.0167* (0.08)	0.92 *0.0121* (0.06)
Normal	Clayton	0.97 *0.0076* (0.31)	1.00 *0.0000* (0.80)	1.00 *0.0000* (0.92)
	Gumbel	0.19 *0.0176* (0.24)	0.13 *0.0151* (0.38)	0.01 *0.0040* (0.38)
	Frank	0.23 *0.0190* (0.08)	0.35 *0.0214* (0.20)	0.60 *0.0219* (0.42)
	Normal	**0.05** *0.0099* **(0.05)**	**0.01** *0.0045* **(0.05)**	**0.00** *0.0000* **(0.05)**
	Student(4)	0.87 *0.0149* (0.10)	0.22 *0.0185* (0.08)	0.08 *0.0120* (0.06)
Student(4)	Clayton	0.71 *0.0204* (0.27)	1.00 *0.000* (0.77)	1.00 *0.000* (0.93)
	Gumbel	0.98 *0.0056* (0.19)	0.74 *0.0331* (0.34)	0.29 *0.0202* (0.42)
	Frank	0.28 *0.4485* (0.09)	0.80 *0.0182* (0.27)	0.02 *0.0061* (0.41)
	Normal	0.84 *0.0166* (0.05)	0.20 *0.0178* (0.04)	0.03 *0.0076* (0.04)
	Student(4)	**0.03** *0.0074* **(0.05)**	**0.01** *0.0034* **(0.05)**	**0.00** *0.0000* **(0.05)**

Table 1: $n_{MC} = 500, n_B = 20, n = 2048, nn = 2048$. Seed 1. Empirical power for the test of $H_0 : c = c_{\lambda_0}$ at the given level $\alpha = 10\%$ where c_{λ_0} is specified in the first column and the data are issue from a copula density specified in the second column. The parameter of each copula density is chosen such that the Kendall's tau is respectively $\tau = 0.25, 0.50, 0.75$.

Family	parameter grid	Cardinal	$\hat{\alpha}$	Decision
Gumbel	$1.05:0.1:1.95$	10	0.00	Yes
Gaussian	$0.0:0.1:0.9$	10	0.04	Yes
Clayton	$0.5:0.1:1.4$	10	0.42	Yes
Frank	$1.5:0.5:6.0$	10	1.00	No
Gumbel	$1.0:0.05:1.95$	20	0.00	Yes
Gaussian	$0.0:0.05:0.95$	20	0.00	Yes
Clayton	$0.5:0.05:1.45$	20	0.54	No
Frank	$1.25:0.25:6.0$	20	1.00	No
Gumbel	1.45	1	0.10	Yes
Gaussien	0.48	1	0.12	Yes
Clayton	0.92	1	0.86	No
Frank	3.20	1	1.00	No
Gumbel	1.36 (1.07%)	1	0.02	Yes
Gaussien	0.45 (3.27%)	1	0.08	Yes
Clayton	0.41 (13.15%)	1	0.62	No
Frank	2.88 (3.93%)	1	1.00	No

Table 2: Empirical probability $\hat{\alpha}$ to reject the fit to a fixed parametrical family given in the first column and Decision at the prescribed level $\alpha = 5\%$. Multivariate null hypotheses (first and second part); $H_0 : c = c_{\hat{\lambda}}$, where $\hat{\lambda}$ is obtained by inversion of the empirical Kendall's tau (third part); $H_0 : c = c_{\tilde{\lambda}}$, where $\tilde{\lambda}$ is obtained by minimizing the ASE quantity which is given into brackets (fourth part).